Macroscopic scale and phase transitions for the Persistent Turning Walker with Alignment model

Charles Elbar
supervised by Pierre Degond, Antoine Diez and Amic Frouvelle

May - July 2019

Abstract

This paper is the result of a two months internship at Imperial College London supervised by Pierre Degond, Antoine Diez and Amic Frouvelle. We modify the PTWA model introduced in [3] and consider a not normalized force acting on the individuals. We investigate the phase transitions that rise from this modification and study their dynamics.

Contents

1 Introduction

Flocks of birds, schools of fish and more generally collective behaviours in animal societies have recently drawn the attention of many fields including mathematics, physics and biology. There are many approaches to study the behaviour of agents with local interactions that led to different types of results. One can for example find a biological approach in [9], [10], [1], [11] and a physical approach in [2]. This paper is concerned with a mathematical point of view. We translate the problem of finding the behaviour at large scale of N agents into the research for a probability distribution that satisifies a Fokker-Planck equation. This method allows us to find rigorous proofs and to tackle questions of stability. The main part of this paper is devoted to the 2 dimensional case but an idea of the model in greater dimensions can be find in the last sections. In the first part of this paper we present the PTWA model and introduce the modifications made in order to get our main model. The modifications are mainly the suppression of the force's normalization. This seemingly minor modification leads to an appearance of phase transitions that depend on a threshold linked with the density of individuals. Then we consider the large time and space scale dynamics of a two-dimensional particle subject to our model. This is done with a hydrodynamic scaling. This scaling depends on a parameter ε that describes the ratio of the microscopic scale to the macroscopic scale. The second part is devoted to the complete study of the model when $\varepsilon \to 0$. We find that the probability distribution converges to an equilibrium which depends on its density. The third part of this paper is concerned with the stability of these equilibria. Finally we suggest the generalization of the model in other dimensions.

2 Presentation of the model

2.1 The PTWA model

The PTWA model (Persistent Turning Walker with Alignment) has been introduced in [3] as a combination of the Cousin-Vicsek alignment model and the PTW model. Cousin-Vicsek algorithm is used to represent the interactions between individuals in animal societies. If we denote by k the k-th particle, n the time, and Δt the time step and X_k^n its position this algorithm reads

$$X_k^{n+1} = X_k^n + c\omega_k^n \Delta t$$

ω_k^n represents the velocity of the individual, belong to the unit sphere and is updated in the following way :

$$\omega_k^{n+1} = \hat{\omega}_k^n$$

where $\hat{\omega}_k^n$ is a random variable on the sphere centered at $\bar{\omega}_k^n = \frac{J_k^n}{|J_k^n|}$ and $J_k^n = \sum_{j,|X_i - X_j| \leq R} \omega_j^n$. The authors from [4] then introduce a frequency of interaction ν and add a Brownian motion B_t in order to find the continuous dynamic system (i.e when $\Delta t \to 0$) :

$$\frac{dX_k}{dt} = c\omega_k$$
$$d\omega_k = (Id - \omega_k \otimes \omega_k)(\nu \bar{\omega}_k dt + \sqrt{2D} dB_t)$$

The integrals should be understood in the Stratonovich sense. The matrix $Id - \omega \otimes \omega$ simply represents the orthogonal projector onto the hyperplane orthogonal to ω. These equations can be interpreted as a competition between the trend to alignment modeled by the first term of the second equation and randomness due to the Brownian motion.
The idea from [3] was to combine this algorithm with the PTW model. The PTW (Persistent Turning Walker) model was introduced to approximate the displacement of an individual fish (with no alignment interaction) and is based on empirical results (see [8]). The main idea is that the fish evolves with a speed of constant norm but with random changes of curvature.

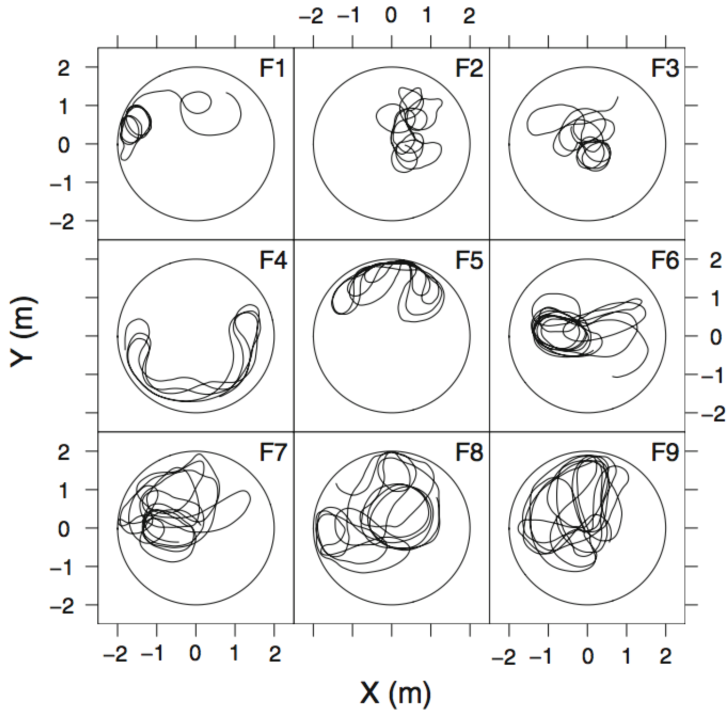


FIGURE 1 – Image and caption taken from [8]. Nine fish trajectories. The trajectories are displayed ranked by the fish speed, from the slowest (fish 1, mean speed 0.16 m/s) to the fastest (fish 9, mean speed 0.56 m.s). The outer circle indicates the tank wall and axis units are in meters.

These trajectories lead the authors from [8] to a system of differential equations for the fish trajectories which is an Ornstein-Uhlenbeck process and reads :

$$\begin{aligned}
&\frac{d\vec{x}}{dt} = c\vec{v}(\theta)\\
&\frac{d\theta}{dt} = c\gamma\\
&d\gamma = -a\gamma dt + b\,dB_t
\end{aligned}$$

which is a two dimensional result. $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$ represents the center of the fish. $\vec{v}(\theta) = (\cos\theta, \sin\theta)$ is the director of the velocity vector and $\theta \in (-\pi, \pi]$ with θ measured from x_1 direction. $\gamma \in \mathbb{R}$ is the curvature of the trajectory and dB_t is the standard Brownian motion. c is the constant magnitude of the velocity. a and b are two constant that represent respectively the relaxation frequency and the intensity of the random curvature jumps.

Combining these two models, the authors from [3] suggested a system of differential equations called the PTWA model with a population of N agents (fish for instance) and the motion of the i^{th} individual is given by :

$$\frac{d\vec{x}_i}{dt} = c\vec{v}(\theta_i) \tag{2.1}$$

$$\frac{d\theta_i}{dt} = c\gamma_i \tag{2.2}$$

$$d\gamma_i = a(\nu\bar{\gamma}_i - \gamma_i)dt + bdB_t^i \tag{2.3}$$

with

$$\bar{\gamma}_i = \vec{v}(\theta_i) \times \overline{\Omega}_i$$

and

$$\overline{\Omega}_i = \frac{J_i}{|J_i|}, \qquad J_i = c \sum_{j=1\ldots N, |\mathbf{x}_i - \mathbf{x}_j| < R} \vec{v}(\theta_j)$$

where the notations are the same than in the PTW model. ν is the typical value of the individuals' trajectory curvature. $\nu\bar{\gamma}_i$ (with $\bar{\gamma}_i$ being a cross product) represents the curvature to achieve for the indidual to align with his neighbors. In order to illustrate this model, we show an image from the article [3] representing a fish adjusting his direction (after a time of order $1/a$) in order to align with his neighbors with direction $\overline{\Omega}$. The curvature is then approximately equal to $\nu\bar{\gamma}$.

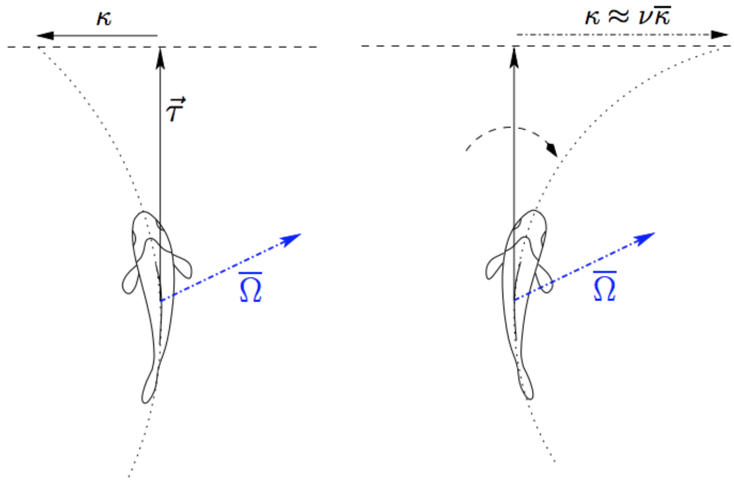


FIGURE 2 – Image and caption taken from [3]. Illustration of the model (2.1)-(2.3). On the left figure, a fish is turning to the left, while its neighbors are moving to the right ($\overline{\Omega}$). After a certain time of order $1/a$, the fish adjusts its curvature in order to align its velocity with $\overline{\Omega}$ (right figure).

Reading the equations, we find a competition between the alignment with neighbors and the tendency for a fish to have a random curvature change (due to the brownian motion). This last behavior is interpreted as a fact that the fish has a tendency to explore the horizons. The main problem in the PTWA model is to study the macroscopic limit which has been treated in [3]. After scaling the variables so that displacement is of the order of the interactions the purpose is to study this model for large scales and time which corresponds to the time and scale of an experiment. We shall describe this idea in the next subsection.

2.2 Main model

In the PTWA model, we notice the normalization of the J_i vector to $\overline{\Omega}_i = \frac{J_i}{|J_i|}$. Thus the interaction term (i.e the first term of 2.3) cannot be recast as a sum of binary interaction and has a singularity when $J_i \to 0$. In this case, we find that the particle distribution (that we shall introduce later) converges to an equilibrium with no phase transitions. Taking off the normalization has led to results with phase transitions which means that the fish can both be aligned or follow a uniform law of probability depending on some parameters. Indeed in [6], the authors investigate the system of self-propelled particles with alignment interactions and show that phase transitions appear depending on the density of individuals using results from [7]. It is important to note that the model from [6] is close to our model, the difference being that in their model the interaction acts directly on the velocity itself whereas in our model the interaction acts on the curvature. We now provide the equations for our model.
We consider first the equations (2.1) to (2.3) and

$$\bar{\gamma}_i = \vec{v}(\theta_i) \times J_i \qquad J_i = \frac{1}{N} \sum_{j=1\ldots N, |x_i - x_j| < R} \vec{v}(\theta_j) \tag{2.4}$$

Remark 2.1. Instead of the condition $|x_i - x_j| < R$ it was possible to introduce a collision Kernel Lipschitz and bounded as in [6]. Moreover, we take $c = 1$ in $\mathbf{J}_i$ in order to respect the dimensions (notice that $\bar{\gamma}_i$ is dimensionless).

We now introduced scaled variables. We use as in [3] $x_0 = \nu^{-1}$ as space unit, $t_0 = (c\nu)^{-1}$ as time unit, $\gamma_0 = x_0^{-1} = \nu$ as curvature unit. And we introduce dimensionless time $t' = t/t_0, x' = x/x_0$ and $\gamma' = \gamma/\gamma_0$. In scaled variables we get (omitting the primes) :

$$\frac{d\vec{x}_i}{dt} = \vec{v}(\theta_i) \tag{2.5}$$

$$\frac{d\theta_i}{dt} = \gamma_i \tag{2.6}$$

$$d\gamma_i = \lambda(\bar{\gamma}_i - \gamma_i)dt + \sqrt{2}\alpha dB_t^i \tag{2.7}$$

where $\bar{\gamma}_i$ is defined by (2.4) with $R \cdot \nu$ renamed by R again and where

$$\lambda = \frac{a}{c\nu}, \qquad \alpha^2 = \frac{b^2}{2c\nu^3}$$

The two coefficients quantify the strength of the interaction compared to random motions. We now introduce the probability distribution $f(t, \vec{x}, \theta, \gamma)d\vec{x}d\theta d\gamma$ which gives the probability to find a fish at time t in a small neighborhood of position $\vec{x}$, velocity angle θ and curvature γ. This type of reasoning is often seen in the domain of kinetic theory where the population is supposed to be large enough i.e $N \to \infty$. Following [3] (see section 3) and using (2.5)-(2.7) we have the following equation for f (dropping vector's arrow for clarity) :

$$\partial_t f + v(\theta) \cdot \nabla_x f + \gamma \partial_\theta f + \lambda \partial_\gamma \big[(\bar{\gamma} - \gamma) f\big] = \alpha^2 \partial_\gamma^2 f \tag{2.8}$$

with

$$\bar{\gamma} = v(\theta) \times J(x) \tag{2.9}$$

and

$$J(x) = \int_{|x-y|<R, \theta, \gamma} v(\theta) f(y, \theta, \gamma) dy d\theta d\gamma \tag{2.10}$$

We can notice that this model is a two-dimensional problem. Another model which is 3 dimensional has been proposed in [2] where rotational symmetries and conservation laws allow the authors to tackle down the problem in a physics way and consider a network of neighbor without a collision kernel.

Now in order to simplify our future model we introduce new notations. Writing $f(t,x,\theta,\gamma) = f_0\tilde{f}(t/\tilde{t_0}, x, \theta, \gamma/\tilde{\gamma_0})$ we get (dropping the tildes) :

$$\partial_t f + t_0 v(\theta)\cdot\nabla_x f = -t_0\gamma_0\gamma\partial_\theta f - \lambda t_0 f_0 (v(\theta)\times J_f)\partial_\gamma f + \lambda t_0 \partial_\gamma(\gamma f) + \alpha^2\frac{t_0}{\gamma_0^2}\partial_{\gamma^2} f$$

With

$$\gamma_0 = \frac{\alpha}{\sqrt{\lambda}} \qquad f_0 = \frac{\alpha}{\lambda^{3/2}}$$

we find

$$\partial_t f + t_0\vec{v}(\theta)\cdot\nabla_x f = \beta_1\Big(-\gamma\partial_\theta f - \bar{\gamma}\partial_\gamma f\Big) + \beta_2\Big(\partial_\gamma(\gamma f) + \partial_{\gamma^2} f\Big) \tag{2.11}$$

and $\beta_1 = \frac{t_0\alpha}{\sqrt{\lambda}}$ or $\beta_2 = \lambda t_0$ can be equal to one depending on the value we choose for t_0. We are interested in the problem (2.11) with (2.9), (2.10).

3 The macroscopic limit

3.1 Hydrodynamic scaling

We are interested at observing the system at large scales. For this purpose we introduce a small parameter ε and the new variables $x' = \varepsilon x$ and $t' = \varepsilon t$. It is important to note that we introduce a linear relationship between the new and old variables. Had we thought the asymptotic regime would be of diffusive nature we would have introduced a quadratic relationship as in [5]. We also introduce $f^\varepsilon(t', x', \theta, \gamma) = \frac{1}{\varepsilon^2} f(t,x,\theta,\gamma)$. The factor $1/\varepsilon^2$ in front of f is not necessary but allow us to keep the same number of individuals. After omitting the primes and inserting f^ε in (2.11) we get :

$$\varepsilon(\partial_t f^\varepsilon + t_0 v(\theta)\cdot\nabla_x f^\varepsilon) + \beta_1\Big(\gamma\partial_\theta f^\varepsilon + \bar{\gamma}^\varepsilon\partial_\gamma f^\varepsilon\Big) = \beta_2\Big(\partial_\gamma(\gamma f^\varepsilon) + \partial_\gamma^2 f^\varepsilon\Big) \tag{3.1}$$

where

$$\bar{\gamma}^\varepsilon = v(\theta)\times J^\varepsilon_{f^\varepsilon}(x) \tag{3.2}$$

and

$$J^\varepsilon_{f^\varepsilon}(x) = \int_{|x-y|<\varepsilon R,\theta,\gamma} v(\theta) f^\varepsilon(y,\theta,\gamma)\frac{dy}{\varepsilon^2}d\theta d\gamma \tag{3.3}$$

Lemma 3.1. *We have the expansion :*

$$J^\varepsilon_{f^\varepsilon}(t,x) = J_{f^\varepsilon}(t,x) + O(\varepsilon^2) \tag{3.4}$$

where the local flux J_f is defined by

$$J_f(t,x) = \int_{\theta,\gamma} v(\theta) f(x,\theta,\gamma) d\theta d\gamma \tag{3.5}$$

Before getting down to the proof of Lemma 3.1 we provide an explanation to this result. In (3.3) we notice that the radius of the interaction ball is of order εR. This means that the interaction radius gets smaller as we look at larger scales i.e the radius is tied to the microscopic scale. This leads us to understand that the information available to the individual is local and thus up to order 1 in ε the equation becomes local.

Proof. Let us expand f^ε at second order in ε assuming it is smooth enough. $\forall(x, y, \theta, \gamma) \in \mathbb{R}^2 \times \mathbb{R}^2 \times (-\pi, \pi] \times \mathbb{R}$, $\exists \xi \in [0, 1]$ such as

$$f^\varepsilon(y, \theta, \gamma) = f^\varepsilon(x, \theta, \gamma) + (y - x) \cdot \nabla_x f^\varepsilon(x, \theta, \gamma) + \frac{1}{2}(y - x)^T \mathbb{H}_x f^\varepsilon(x + \xi(y - x), \theta, \gamma)(y - x)$$

where $\mathbb{H}_x$ represents the Hessian and $(y - x)^T$ the transpose of $(y - x)$. We insert this formula into the integral (3.3) and use the change of variable $y = x + \varepsilon u$. We get

$$J^\varepsilon_{f^\varepsilon}(t, x) = \int_{|u|<R} du \int_{\theta,\gamma} v(\theta) f^\varepsilon(x, \theta, \gamma) d\theta d\gamma + \int_{|u|<R,\theta,\gamma} v(\theta)\varepsilon u \cdot \nabla_x f^\varepsilon(x, \theta, \gamma) d\theta d\gamma + O(\varepsilon^2)$$

where $O(\varepsilon^2)$ is obtained assuming f^ε and its derivatives are correctly bounded. The second term vanishes as $\int_{|u|<R} u du = 0$. Finally we normalize the coefficient $\int_{|u|<R} du$ which is the volume of unit ball in dimension 2 to be 1 by modifying the collision kernel. The lemma follows. □

Equation (3.1) becomes

$$\varepsilon(\partial_t f^\varepsilon + t_0 v(\theta) \cdot \nabla_x f^\varepsilon) = Q(f^\varepsilon) + O(\varepsilon^2) \tag{3.6}$$

with the collision operator Q defined by

$$Q(f) = \beta_1\Big(- \gamma\partial_\theta f - \bar{\gamma}^\varepsilon \partial_\gamma f\Big) + \beta_2\Big(\partial_\gamma(\gamma f) + \partial_{\gamma^2} f\Big) \tag{3.7}$$

with

$$\bar{\gamma}^\varepsilon = v(\theta) \times J_{f^\varepsilon}$$

Notice that the collision operator acts only on the variables θ and γ and leave the other variables t and x as parameters. Hence the idea is to study Q as an operator acting on θ and γ only. We remind that the main goal is to determine the model's limit when $\varepsilon \to 0$. With that in mind we suppose for instance that f^ε converges to an equilibrium f. Then the left hand side would vanish when $\varepsilon \to 0$. This would leave us with $Q(f^\varepsilon) = 0$. Therefore we want to study the equilibria i.e the functions f such that $Q(f) = 0$.

4 The collision operator

4.1 Equilibria

In order to study the equilibria as in [3] we define $\bar{\theta}$ such that

$$v(\bar{\theta}) = \Omega_{f^\varepsilon} = \frac{J_{f^\varepsilon}}{|J_{f^\varepsilon}|}$$

We get

$$v(\theta) \times J_{f^\varepsilon} = |J_{f^\varepsilon}| \sin(\bar{\theta} - \theta)$$

And thus

$$Q(f) = \beta_1\Big(- \gamma\partial_\theta f - |J_f|\sin(\bar{\theta} - \theta)\partial_\gamma f\Big) + \beta_2\Big(\partial_\gamma(\gamma f) + \partial^2_\gamma f\Big) \tag{4.1}$$

We also define the density ρ_f associated to f by :

$$\rho_f(x,t) = \int_{\theta,\gamma} f(t,x,\theta,\gamma)d\theta d\gamma \tag{4.2}$$

Notice that the collision operator can be decomposed in two parts : a skew-ajoint operator (i.e $\langle Tf, g\rangle = -\langle f, Tg\rangle$ where T is the operator) which is $-\gamma\partial_\theta f - |J_f|\sin(\bar{\theta} - \theta)\partial_\gamma f$ and a self-adjoint operator (not for the same dot product) : $\partial_\gamma(\gamma f) + \partial^2_\gamma f$. The main difference with the equilibria of [3] is that our equilibria will be find implicitly given. For the skew-adjoint part, we introduce the function :

$$H^{J_f}(\theta,\gamma) = -|J_f|\cos(\theta) + \frac{\gamma^2}{2}$$

and we adopt the convention that for any function $h(\theta,\gamma)$:

$$h_{J_f}(\theta,\gamma) = h_{\bar{\theta}}(\theta,\gamma) = h(\theta - \bar{\theta},\gamma) \tag{4.3}$$

with $v(\bar{\theta}) = \Omega_f$. Notice that :

$$H^{J_f}_{\bar{\theta}} = -J_f \cdot v(\theta) + \frac{\gamma^2}{2} \tag{4.4}$$

This previous remark is not useful for the moment but will be used for the stability section.
Using these notations, for any smooth function f, the skew-adjoint part of Q can be written as :

$$-\beta_1\gamma\partial_\theta f - \beta_1|J_f|\sin(\bar{\theta} - \theta)\partial_\gamma f = \beta_1\big(\partial_\theta H^{J_f}_{\bar{\theta}}\partial_\gamma f - \partial_\gamma H^{J_f}_{\bar{\theta}}\partial_\theta f\big) = \beta_1\left\{H^{J_f}_{\bar{\theta}}, f\right\}_{(\theta,\gamma)} \tag{4.5}$$

using the Poisson Bracket $\{\cdot,\cdot\}_{(\theta,\gamma)}$ formalism in the (θ,γ) space that we write $\{\cdot,\cdot\}$ from now on for sake of clarity. Therefore any function of the form $g(H_{\bar{\theta}})$ satisfies $\{H_{\bar{\theta}}, g(H_{\bar{\theta}})\} = 0$.

On the other hand, the self adjoint part of Q satisfies :

$$\beta_2\partial_\gamma(\gamma f) + \beta_2\partial^2_\gamma f = \beta_2\partial_\gamma\Big(\mathcal{N}\partial_\gamma\Big(\frac{f}{\mathcal{N}}\Big)\Big) \tag{4.6}$$

where $\mathcal{N}$ is the Gaussian distribution with zero mean and variance 1 :

$$\mathcal{N}(\gamma) = \sqrt{\frac{1}{2\pi}}\exp\Big(-\frac{\gamma^2}{2}\Big) \tag{4.7}$$

In particular, the Gaussian $\mathcal{N}$ is in the kernel of the self-adjoint part of Q. We combine the two previous observations to define the function :

$$\mu^J(\theta,\gamma) = C\exp\Big(-H^J\Big) = C\exp\Big(-\Big(\frac{\gamma^2}{2} - |J|\cos\theta\Big)\Big) \tag{4.8}$$

where C is the normatization constant such that the integral of μ is one. The translates $\mu_{\bar{\theta}}$ of μ in the sense of definition (4.3) are of the form $g(H_{\bar{\theta}})$ and are Gaussian distributions in γ with variance 1. It follows from a simple computation that $\mu_{\bar\theta}$ is an equilibrium for Q for all real values of $\bar\theta$.

We define the Von Mises distribution $\mathcal{M}$:

$$\mathcal{M}^J(\theta) = C_0 \exp\Big(|J|\cos\theta\Big) \tag{4.9}$$

where $C_0 = \big(2\pi I_0(|J|)\big)^{-1}$ represents the normalization such that $\int_\theta M^J(\theta)d\theta = 1$ with I_0 the modified Bessel function of order 0. Note that in comparison to [3] we need to let $\mathcal{M}$ depend on J, which will lead to phase transitions. We can write μ as a product of $\mathcal{M}$ given by (4.9) and $\mathcal{N}$ given by (4.7).

$$\mu^J(\theta,\gamma) = \mathcal{M}^J(\theta)\mathcal{N}(\gamma) \tag{4.10}$$

The properties of Q are summarized in the following proposition :

Proposition 4.1. *i) The operator Q satisfies :*

$$\int_{\theta,\gamma} Q(f)\frac{f}{\mu_{\bar\theta}^{J_f}}d\theta d\gamma = -\beta_2\int_{\theta,\gamma}\frac{\mathcal{N}}{\mathcal{M}_{\bar\theta}^{J_f}}\Big|\partial_\gamma\Big(\frac{f}{\mathcal{N}}\Big)\Big|^2 d\theta d\gamma \le 0 \tag{4.11}$$

with μ defined by (4.8) and $\bar\theta$ such that $v(\bar\theta) = \Omega_f$.

ii) If f is an equilibria of Q (i.e the functions $f(\theta,\gamma)\ge 0$ such that $Q(f)=0$) then f is of the form :

$$f = \rho\mu_{\bar\theta}^{J_f} \quad \textit{with } \rho\in\mathbb{R}^+, \bar\theta\in(-\pi,\pi] \tag{4.12}$$

where ρ is the total mass and $\bar\theta$ the direction of the flux of $\rho\mu_{\bar\theta}$. Note that if we integrate with respect to (θ,γ) then $\rho=\rho_f$ where ρ_f has been defined in (4.2).

Proof. (i) Combining (4.5) and (4.6), we find :

$$Q(f) = \{H_{\bar\theta}, f\} + \beta_2\partial_\gamma\Big(\mathcal{N}\partial_k\Big(\frac{f}{\mathcal{N}}\Big)\Big) \tag{4.13}$$

Using (4.8), the fact that the Poisson bracket with f is a derivation and a skew-adjoint operator, we find :

$$\begin{aligned}\int_{\theta,\gamma}\{H_{\bar\theta}, f\}\frac{f}{\mu}d\theta d\gamma &= \int_{\theta,\gamma}\exp\Big(-H_{\bar\theta}\Big)\Big\{\exp\Big(H_{\bar\theta}\Big), f\Big\}\frac{f}{\mu}d\theta d\gamma\\ &= \frac{1}{C_0}\int_{\theta,\gamma}\Big\{\exp\Big(H_{\bar\theta}\Big), f\Big\} f d\theta d\gamma = 0.\end{aligned}$$

Then using the formulation of Q in (4.13), we easily deduce the equality (4.11) by applying Green's formula.

(ii) If f is an equilibrium for Q (i.e $Q(f)=0$) using the equality in (4.11) we have f is proportional to $\mathcal{N}$ as a function of γ. Therefore we can write :

$$f(\theta,\gamma) = \varphi(\theta)\mathcal{N}(\gamma)$$

Using that f is an equilibrium, we have :

$$-\gamma\varphi'(\theta) + |J_f|\sin(\bar\theta-\theta)\gamma\varphi(\theta) = 0.$$

Solving this equation leads to $\varphi(\theta) = C\mathcal{M}_{\bar\theta}^{J_f}(\theta)$ with $\mathcal{M}$ given by (4.9). Hence f is of the form $f=\rho\mu_{\theta_0}^{J_f}$, with $\rho\ge 0$ and $\theta_0\in(-\pi,\pi]$.

□

We now have an implicit equation for f when it is an equilibrium. Let's first compute J_f for f given by (4.12).

$$
\begin{aligned}
J_f &= \int_{\theta,\gamma} \rho \mu_{\theta_0}^{J_f} v(\theta) d\theta d\gamma \\
&= \rho \int_{\theta,\gamma} \mathcal{N}(\gamma) C_0 \exp\Big(|J_f| \cos(\theta-\theta_0)\Big) \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} d\theta d\gamma
\end{aligned}
$$

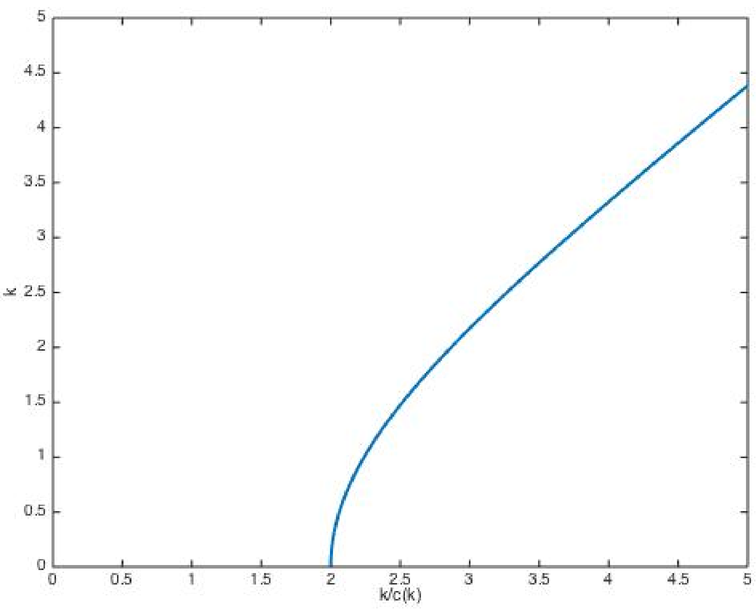
Then by the change of variables $\phi = \theta - \theta_0$ and using oddness considerations, we obtain

$$
J_f = \rho C_0 \int_\phi \exp\Big(|J_f|\cos(\phi)\Big)\cos(\phi) d\phi v(\bar\theta) = \rho \frac{I_1(|J_f|)}{I_0(|J_f|)} v(\bar\theta)
$$

where I_0, I_1 are the modified Bessel function of order 0 and 1 and the fact that the normalization of $C_0 = \frac{1}{2\pi I_0(|J_f|)}$ of $\mathcal{M}^J$ has been used. We notice that $J_f = |J_f| v(\bar\theta)$ which leads to the following equation for $\kappa = |J_f|$:

$$
\kappa = \rho \frac{\int_\phi \exp\Big(\kappa\cos(\phi)\Big)\cos(\phi)d\phi}{\int_\phi \exp\Big(\kappa\cos(\phi)\Big)d\phi} = \rho \frac{I_1(\kappa)}{I_0(\kappa)} = \rho c(\kappa) \tag{4.14}
$$

Let $\kappa \to 0$. As in [6] we are interested in the quantity $\kappa \mapsto \frac{c(\kappa)}{\kappa}$. A simple Taylor expansion in the two integrals allow us to conclude that $\lim_{\kappa\to 0} \frac{c(\kappa)}{\kappa} = \frac{1}{2}$. We also notice that $\kappa \mapsto \frac{c(\kappa)}{\kappa}$ is decreasing. Therefore there is no other solution than $\kappa = 0$ if $\rho \le 2$. On the other hand, if $\rho > 2$ there is a unique strictly positive solution in addition to the trivial solution $\kappa = 0$. This discussion is summarized in the following graph where κ has been plotted in function of $\frac{\kappa}{c(\kappa)}$:



With this reasoning we find the following proposition :

Proposition 4.2. *(i) If $\rho \le 2$, $\kappa = 0$ is the only solution to the compatibility relation (4.14). The only equilibria are the isotropic ones of the form $f = \rho\mathcal{N}(\gamma)/2\pi$ with arbitrary $\rho \ge 0$.*
(ii) If $\rho > 2$, the compatibility relation (4.14) has exactly two roots : $\kappa = 0$ and a unique

strictly positive root denoted by $\kappa(\rho)$. The set of equilibria associated to the root $\kappa = 0$ consist of the isotropic equilibria $f = \rho\mathcal{N}(\gamma)/2\pi$, with arbitrary $\rho > 2$. The set of equilibria associated to the root $\kappa(\rho)$ are of the form $f = \rho\mu_{\bar{\theta}}^{\kappa(\rho)\Omega}$ with arbitrary $\rho > 2$, $\Omega \in \mathbb{S}$ and $\bar{\theta}$ such that $v(\bar{\theta}) = \Omega$.

The proof of this proposition has been provided above. We now make some comments about this result. First of all the factor two may come from the dimension of the problem. Indeed, in [6] where a model of dimension n (acting on the velocities) has been proposed, the threshold found is n. Moreover we prove in the last section that the threshold in the 3 dimensional case for our problem is 3. The second result is that when the force is not normalized phase transitions appear, from isotropic and disordered states at low density to more aligned states at high densities.

5 Taking the limit $\varepsilon \to 0$

5.1 The disordered and ordered regions

Recall the proposition 4.2 and more precisely the threshold $\zeta = 2$. This threshold leads us to consider two regions in our space so that the solution will converge in a sense defined later differently in those regions. Note that ρ depends on the variables (x, t). We thus define the disordered region $\mathcal{R}_d$ and the ordered region $\mathcal{R}_0$ as :

$$\mathcal{R}_d = \{(x,t) \mid \zeta - \rho_\varepsilon(x,t) \gg \varepsilon, \text{ as } \varepsilon \to 0\} \tag{5.1}$$

$$\mathcal{R}_0 = \{(x,t) \mid \zeta - \rho_\varepsilon(x,t) \ll \varepsilon, \text{ as } \varepsilon \to 0\} \tag{5.2}$$

We now make a hypothesis that allows us to continue our work :

Hypothesis 1. When $\varepsilon \to 0$ we have :

$$f^\varepsilon \to \rho(x,t)\mathcal{N}(\gamma)/2\pi, \quad \forall (x,t) \in \mathcal{R}_d \tag{5.3}$$

$$f^\varepsilon \to \rho(x,t)\mu_{\bar{\theta}}^{\kappa(\rho)\Omega(x,t)}, \quad \forall (x,t) \in \mathcal{R}_0 \tag{5.4}$$

where $\rho(x,t)$ has been defined in (4.2) and we assume that the convergence is as smooth as needed.

Now this hypothesis that also appears in [3] and [6] seems normal but it is in fact a difficult result. But some equivalent has been shown and we refer the reader to [12] for instance and leave that for future work.

In order to completely determine f we need to find equations satisfied by $\rho(x,t)$ and $\Omega(x,t)$. This is the object of the following subsections.

Before getting down to the problem in both regions we first note the mass conservation equation in both regions.

$$\int_{\theta,\gamma} Q(f^\varepsilon)d\theta d\gamma = 0$$

Integrating (3.6) with respect to (θ, γ) leads to the following equation for $\rho^\varepsilon(x,t)$:

$$\partial_t \rho^\varepsilon + \nabla_x \cdot J_{f^\varepsilon} = 0. \tag{5.5}$$

5.2 The order 1 in ε in the disordered region

In the disordered region we have (5.3). Hence $J_{f^\varepsilon} \to \rho(x,t)/2\pi \int_{\theta,\gamma} v(\theta)\mathcal{N}(\gamma)d\theta d\gamma = 0$ and (5.5) leads to

$$\partial_t \rho = 0 \tag{5.6}$$

with ρ being the limit when $\varepsilon \to 0$ of ρ^ε. We obtained what we required in the disordered region but as in [6] we look for more precise information at the next order in ε.
For this purpose we follow the idea of [5] and we notice the following proposition :

Proposition 5.1. *Let $Df = \partial_\gamma(\gamma f) + \partial_{\gamma^2} f$ and let f ang g be smooth functions decreasing at infinity. The following identities hold true :*

$$\begin{aligned} Df &= \partial_\gamma\Big(\mathcal{N}\partial_\gamma\Big(\frac{f}{\mathcal{N}}\Big)\Big) \\ \int_{\mathbb{R}} Dfg\frac{d\gamma}{\mathcal{N}} &= -\int_{\mathbb{R}} \mathcal{N}\partial_\gamma\Big(\frac{f}{\mathcal{N}}\Big)\Big(\frac{g}{\mathcal{N}}\Big)d\gamma = \int_{\mathbb{R}} fDg\frac{d\gamma}{\mathcal{N}} \\ \int_{\mathbb{R}} Dff\frac{d\gamma}{\mathcal{N}} &= -\int_{\mathbb{R}} \mathcal{N}\Big|\partial_\gamma\Big(\frac{f}{\mathcal{N}}\Big)\Big|^2 d\gamma \le 0 \\ Df = 0 &\iff \exists c \in \mathbb{R}, f = c\mathcal{N} \end{aligned}$$

This proposition shows that the natural L^2 norm associated with this operator has a weight $\mathcal{N}^{-1}$. Thanks to this proposition we define the following functional spaces :

$$\begin{aligned} H &= \{u : (-\pi,\pi] \times \mathbb{R}_\gamma \to \mathbb{R} / \int_{\theta,\gamma} |u(\theta,\gamma)|^2 \frac{d\theta d\gamma}{\mathcal{N}} < +\infty\}, \\ V &= \Big\{u \in H / \int_{\theta,\gamma} \mathcal{N}\Big|\partial_\gamma\Big(\frac{u}{\mathcal{N}}\Big)\Big|^2 d\theta d\gamma \le +\infty\Big\}, \\ L^2_{\mathcal{N}} &= L^2(\mathbb{R}^2_x, H), \qquad X = L^2([0,T] \times \mathbb{R}^2_x, V) \end{aligned}$$

Note that we can expect f to decay fast enough to belong to these sets. We also define the operator A :

$$Af = -\partial_\gamma(\gamma f) - \partial_{\gamma^2} f + \beta_1 \gamma \partial_\theta f. \tag{5.7}$$

and its domain :

$$D(A) = \{u(\theta,\kappa) \in V / Au \in H\} \tag{5.8}$$

In the following theorem we will also need this proposition from [5] :

Proposition 5.2. *Let $g \in H$ where H has been defined above. Then, there exists $u \in D(A)$ such that*

$$Au = g$$

if and only if g satisfies the following solvability condition :

$$\int_{\theta,\gamma} g(\theta,\gamma)d\theta d\gamma = 0.$$

Moreover, the solution u is unique up to a constant times $\mathcal{N}$. A unique solution can be singled out by prescribing the condition

$$\int_{\theta,\gamma} u(\theta,\gamma)d\theta d\gamma = 0.$$

These tools allow us to find the following theorem for the computation of f^ε up to order 1 in ε :

Theorem 5.1. *When $\varepsilon \to 0$, the formal (first) order approximation to the solution of the rescaled equation (3.6) for $f^\varepsilon \in D(A)$ in the disordered region defined by (5.1) is given by :*

$$f^\varepsilon(t,x,\theta,\gamma) = \frac{\rho^\varepsilon(x,t)}{2\pi}\mathcal{N}(\gamma) + \varepsilon\Big(- \chi_d \cdot \nabla_x \rho^\varepsilon + \frac{\rho^\epsilon}{2\pi} J_{f_1^\varepsilon} \cdot v(\theta)\mathcal{N}(\gamma)\Big) \tag{5.9}$$

where χ_d has been defined in the following proof and $J_{f_1^\varepsilon}$ satisfies the following equation :

$$J_{f_1} = -\frac{2M \cdot \nabla_x \rho^\varepsilon}{2-\rho^\varepsilon}$$

where $M = \int_{\theta,\gamma} \chi_d d\theta d\gamma$ and the density ρ^ε satisfies :

$$\partial_t \rho^\varepsilon = \varepsilon \nabla_x \cdot \Big(\frac{2M \cdot \nabla_x \rho^\varepsilon}{2-\rho^\varepsilon}\Big). \tag{5.10}$$

Remark 5.1. It is possible to find χ_d such that $M = 0$. This would lead to $f^\varepsilon(t,x,\theta,\gamma) = \frac{\rho^\varepsilon(x,t)}{2\pi}\mathcal{N}(\gamma) - \varepsilon\chi_d \cdot \nabla_x \rho^\varepsilon$.

Proof. We write $f^\varepsilon = \rho^\varepsilon(x,t)\mathcal{N}(\gamma)/2\pi + \varepsilon f_1^\varepsilon(t,x,\theta,\gamma)$. Recall (4.2) and the definition of $\mathcal{N}$. Integrating with respect to (θ,γ) gives $\int_{\theta,\gamma} f_1^\varepsilon d\theta d\gamma = 0$.

We also have :

$$J_{f^\varepsilon} = \varepsilon J_{f_1^\varepsilon} \tag{5.11}$$

With

$$Q(f) = \beta_1\Big(- \gamma\partial_\theta f - \bar{\gamma}^\varepsilon \partial_\gamma f\Big) + \beta_2 \partial_\gamma \Big(\mathcal{N}\partial_k\Big(\frac{f}{\mathcal{N}}\Big)\Big)$$

we find using (3.6) :

$$\begin{aligned} &\partial_t \rho^\varepsilon \mathcal{N}(\gamma)/2\pi + t_0 v(\theta)\cdot\nabla_x\rho^\varepsilon\mathcal{N}(\gamma)/2\pi + \varepsilon(\partial_t f_1^\varepsilon + t_0 v(\theta)\cdot\nabla_x f_1^\varepsilon) = \\ &\qquad - \beta_1\gamma\partial_\theta f_1^\varepsilon - \varepsilon\beta_1\bar{\gamma}_1^\varepsilon\partial_\gamma f_1^\varepsilon + \beta_2\partial_\gamma\Big(\mathcal{N}\partial_\gamma\Big(\frac{f_1^\varepsilon}{\mathcal{N}}\Big)\Big) + \beta_1\bar{\gamma}_1^\varepsilon\gamma\mathcal{N}(\gamma)\rho^\varepsilon/2\pi \end{aligned} \tag{5.12}$$

with $\bar{\gamma}_1^\varepsilon = v(\theta) \times J_{f_1^\varepsilon}$

From (5.5) and using (5.11) we have $\partial_t \rho^\varepsilon = O(\varepsilon)$. In order to find f_1^ε we may only retain the terms of order 0 in (5.12). We get

$$\beta_2\partial_\gamma\Big(\mathcal{N}\partial_\gamma\Big(\frac{f_1^\varepsilon}{\mathcal{N}}\Big)\Big) = t_0 v(\theta)\cdot\nabla_x\rho^\varepsilon\mathcal{N}(\gamma)/2\pi - \beta_1\bar{\gamma}_1^\varepsilon\gamma\mathcal{N}(\gamma)\rho^\varepsilon/2\pi + \beta_1\gamma\partial_\theta f_1^\varepsilon + O(\varepsilon) \tag{5.13}$$

Now we make the choice $\beta_2 = 1$ (choosing the value of $t_0 = \frac{1}{\lambda}$ until the end) and we write (dropping the ε on f_1^ε for the sake of clarity) :

$$\partial_\gamma(\gamma f_1) + \partial_{\gamma^2} f_1 - \beta_1\gamma\partial_\theta f_1 = t_0 v(\theta)\cdot\nabla_x\rho^\varepsilon\mathcal{N}(\gamma)/2\pi - \beta_1\bar{\gamma}_1^\varepsilon\gamma\mathcal{N}(\gamma)\rho^\varepsilon/2\pi + O(\varepsilon) \tag{5.14}$$

We are looking for solutions of the form :

$$f_1(\theta,\gamma) = f_2(\gamma,\theta) + \varphi(\theta)\mathcal{N}(\gamma)$$

where φ is a fixed function that we choose so that the the second term of the right hand side vanishes. We get :

$$\varphi(\theta) = \frac{\rho^\varepsilon}{2\pi} J_{f_1} \cdot v(\theta) \tag{5.15}$$

It only remains to find f_2 which satisfies :

$$\partial_\gamma(\gamma f_2) + \partial_{\gamma^2} f_2 - \beta_1 \gamma \partial_\theta f_2 = t_0 \mathcal{N}(\gamma)\nabla_x \rho^\varepsilon \cdot v(\theta)/2\pi. \tag{5.16}$$

(5.16) can be written as

$$-Af_2 = t_0 \mathcal{N}(\gamma)\nabla_x \rho^\varepsilon \cdot v(\theta)/2\pi.$$

Using proposition 5.2 with $g = \frac{t_0}{2\pi}\mathcal{N}(\gamma)v(\theta) \in H$ we find χ_d a solution associated to $Au = g$ and we get $f_2 = -\chi_d \cdot \nabla_x \rho^\varepsilon$

Now using (5.15) :

$$f_1 = -\chi_d \cdot \nabla_x \rho^\varepsilon + \frac{\rho^\epsilon}{2\pi} J_{f_1} \cdot v(\theta)\mathcal{N}(\gamma)$$

Multiplying by $v(\theta)$ and intgrating with respect to (θ,γ) yields :

$$J_{f_1} = -M \cdot \nabla_x \rho^\varepsilon + \frac{\rho^\epsilon}{2} J_{f_1} \tag{5.17}$$

where M is a constant vector with respect to (θ,γ). This leads to

$$J_{f_1} = -\frac{2M \cdot \nabla_x \rho^\varepsilon}{2-\rho^\varepsilon}$$

Reinserting this expression into the expression of f_1 completes the proof. □

Remark 5.2. We note the presence of the term $(2-\rho^\varepsilon)$ in the denominator of f^ε. Indeed this approximation is only valid in the disordered region and blows up as $\rho \to 2$.

5.3 The ordered region

In this region we have (5.4)

Theorem 5.2. *When $\varepsilon \to 0$, the formal limit to the solution $f^\varepsilon(t,x,\theta,\gamma)$ of (3.6), in the ordered region $\mathcal{R}_0 \subset \mathbb{R}^2$ defined by (5.2) is given by*

$$f(t,x,\theta,\gamma) = \rho(x,t)\mu_{\bar\theta}^{\kappa(\rho(x,t))\Omega(x,t)}(\theta,\gamma) \tag{5.18}$$

where $\mu_{\bar\theta}^{\beta\Omega}$ is defined by (4.10), $\bar\theta$ such as $v(\bar\theta) = \Omega_f$ and β is the unique positive solution to the compatibility condition (4.14). Moreover the density $\rho > \zeta$ and the orientation $\Omega \in \mathbb{S}^1$ satisfy the following system of first order partial differential equations :

$$\partial_t \rho + \nabla_x \cdot (c_1 \rho \Omega) = 0 \tag{5.19}$$

$$\rho\partial_t \Omega + c_2 \rho(\Omega \cdot \nabla_x)\Omega + c_3(Id - \Omega \otimes \Omega)\nabla_x \rho + c_4 \rho(Id - \Omega \otimes \Omega)\nabla_x \rho = 0 \tag{5.20}$$

where the coefficient $c_1 = c_1(\kappa(\rho))$ is defined by (4.14) and the coefficients c_2, c_3, c_4 are defined by (5.39), (5.40) and (5.41).

We prove this theorem in the subsection entitled proof of theorem 5.2. In order to prove it, we need two conservation equations. However, other than the mass conservation combined with the compatibility relation which lead to (5.19) there is no other obvious conservation equations. To completely determine our system i.e find the density ρ and our unit vector Ω we have to look at the collisional invariants. This is the object of the following subsection.

5.4 Generalized collisional invariant

We introduce as in [3] the generalized collisional invariant. A collision invariant of the operator Q is a function ψ which satisfies :

$$\int_{\theta,\gamma} Q(f)\psi d\theta d\gamma = 0, \quad \text{for all } f.$$

There is no obvious collisional invariant apart from the constants. Hence we introduce the following definition :

Definition 5.1. For a given $J = \kappa\Omega$ and a given distribution $f(\theta,\gamma)$, we define the linear 'extended' collision operator $\mathcal{Q}_{\kappa\Omega}(f)$ by :

$$\mathcal{Q}_{\kappa\Omega}(f) = \{H_{\kappa\Omega}, f\} + \beta_2\partial_\gamma\Big(\mathcal{N}\partial_k\Big(\frac{f}{\mathcal{N}}\Big)\Big)$$

where the notations are that of (4.3). We note that

$$Q(f) = \mathcal{Q}_{J_f}(f)$$

The definition of a Generalized Collision Invariant is the following one :

Definition 5.2. For a given vector $J = \kappa\Omega$, a function $\psi_{\kappa\Omega}$ is called a Generalized Collisional Invariant (GCI) if it satisfies :

$$\int_{\theta,\gamma} \mathcal{Q}_{\kappa\Omega}(f)\psi_{\kappa\Omega} d\theta d\gamma = 0, \quad \forall f \text{ such that } J_f \times \Omega = 0$$

We note the following adjoint operator of $\mathcal{Q}_{\kappa\Omega}$:

$$\mathcal{Q}^*_{\kappa\Omega}(\psi) = \beta_1(\gamma\partial_\theta\psi + \kappa\sin(\bar\theta - \theta)\partial_\gamma\psi) - \gamma\partial_\gamma\psi + \partial_{\gamma^2}\psi$$

with $\bar\theta$ such that $\Omega = v(\bar\theta)$. This operator gives us the following lemma for a GCI $\psi_{\kappa\Omega}$ where a proof can be found in [3].

Lemma 5.1. *For given $(\kappa,\Omega) \in \mathbb{R} \times \mathbb{S}^1$, a function $\psi_{\kappa\Omega}$ is a generalized collisional invariant if and only if there exists a constant $\beta \in \mathbb{R}$ such that :*

$$\mathcal{Q}^*_{\kappa\Omega}(\psi) = \beta\vec{v}(\theta) \times \Omega \tag{5.21}$$

Following [3] we use the notations :

$$L^2_\mu = \{f(\theta,\gamma)/\int_{\theta,\gamma} |f|^2\mu d\theta d\gamma < +\infty\}, \tag{5.22}$$

$$\langle f, g\rangle_\mu = \int_{\theta,\gamma} fg\mu d\theta d\gamma. \tag{5.23}$$

$$\langle g\rangle_\mu = \int_{\theta,\gamma} g\mu d\theta d\gamma. \tag{5.24}$$

We also define the hyperplane E :

$$E = \{f \in L^2_\mu / \int_{\theta,\gamma} f\mu d\theta d\gamma = 0\}$$

and the linear operator $\mathcal{L}$:

$$\mathcal{L}_\kappa \psi = \beta_1(\gamma \partial_\theta \psi - \kappa \sin(\theta)\partial_\gamma \psi) - \gamma \partial_\gamma \psi + \partial_{\gamma^2} \psi \tag{5.25}$$

with domain $D(\mathcal{L})$ given by :

$$D(\mathcal{L}_\kappa) = \{f \in L^2_\mu / \mathcal{L}_\kappa f \in L^2_\mu\}$$

We also get from [3] the following lemma and proposition :

Lemma 5.2. *(i) Let $\chi \in L^2_\mu$. A necessary condition for the existence of a solution $\psi \in D(\mathcal{L}_\kappa)$ of problem*

$$\mathcal{L}_\kappa \psi = \chi \tag{5.26}$$

is that $\chi \in E$.

(ii) For all $\chi \in E$, the problem (5.26) has a unique solution ψ in E. Then all solutions to problem (5.26) are of the form $\psi + K$ with an arbitrary $K \in \mathbb{R}$.

Proposition 5.3. *For given $(\kappa, \Omega) \in \mathbb{R} \times \mathbb{S}^1$, the cet $C_{\kappa\Omega}$ of the GCI's associated to Ω is a two dimensional vector space $C_{\kappa\Omega} = Span\{1, \psi_\Omega\}$ where ψ_Ω is given by :*

$$\psi_{\kappa\Omega}(\theta, \gamma) = \psi(\theta - \bar{\theta}, \gamma) \tag{5.27}$$

with $\bar{\theta}$ such that $v(\bar{\theta}) = \Omega$ and ψ is the unique solution of :

$$\mathcal{L}_\kappa \psi = -sin(\theta), \tag{5.28}$$

belonging to the hyperplane E. Moreover, the function ψ satisfies the property :

$$\psi(-\theta, -\gamma) = -\psi(\theta, \gamma) \tag{5.29}$$

5.5 Proof of theorem 5.2

Proof. Recall that in the ordered region we have (5.4). Now thanks to the compatibility relation : $J = \rho c_1 \Omega$ where $c_1 = c_1(\kappa(\rho))$ is defined by (4.14). Therefore using (5.5) in the limit $\varepsilon \to 0$ we get :

$$\partial_t \rho + \nabla_x \cdot (\rho c_1 \Omega) = 0.$$

In order to fully determine our system we use the generalized collisional invariants. We integrate (3.6) against the generalized collisional invariant $\psi_{\kappa^\varepsilon \Omega^\varepsilon}$ defined by (5.27). This leads to

$$\int_{\theta,\gamma} (\partial_t f^\varepsilon + t_0 \vec{v}(\theta) \cdot \nabla_x f^\varepsilon) \psi_{\kappa^\varepsilon \Omega^\varepsilon} d\theta d\gamma = 0.$$

In the ordered region we have $f^\varepsilon \to \rho(x,t)\mu_{\bar{\theta}}^{\kappa(\rho)\Omega(x,t)}$.

In the limit $\varepsilon \to 0$ we find :

$$\int_{\theta,\gamma} \partial_t(\rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega}\mathcal{N})\psi_{\kappa\Omega}d\theta d\gamma + \int_{\theta,\gamma} t_0 v(\theta)\cdot\nabla_x(\rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega}\mathcal{N})\psi_{\kappa\Omega}d\theta d\gamma = 0. \tag{5.30}$$

Using the chain rule and (4.9) we get :

$$\partial_t(\rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega}) + t_0 v(\theta)\cdot\nabla_x(\rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega}) = \partial_t\rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega} + \rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega}\kappa\sin(\theta-\bar\theta)\partial_t\bar\theta$$
$$-\rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega}C_0\dot\kappa\partial_t\rho I_1(\kappa) - t_0\rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega}C_0\dot\kappa v(\theta)\cdot\nabla_x\rho I_1(\kappa) + \rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega}\dot\kappa\cos(\theta-\bar\theta)\partial_t\rho + t_0 v(\theta)\cdot(\nabla_x\rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega})$$
$$+\rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega}\kappa\sin(\theta-\bar\theta)\nabla_x\bar\theta) + t_0 v(\theta)\cdot(\rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega}\dot\kappa\cos(\theta-\bar\theta)\nabla_x\rho))$$

with $C_0 = I_0(\kappa)^{-1}$ and where $\dot\kappa$ is the derivative of κ with respect to the variable ρ and I_1 the modified bessel function

Therefore equation (5.30) leads to :

$$\begin{aligned}
&\int_{\theta,\gamma} \partial_t\rho\mathcal{M}_{\bar\theta}\mathcal{N}\psi_{\bar\theta}d\theta d\gamma\\
&+\kappa\int_{\theta,\gamma} \rho\mathcal{M}_{\bar\theta}\mathcal{N}\sin(\theta-\bar\theta)\partial_t\bar\theta\psi_{\bar\theta}d\theta d\gamma\\
&-\dot\kappa\int_{\theta,\gamma} \rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega}C_0\partial_t\rho I_1(\kappa)\psi_{\bar\theta}d\theta d\gamma\\
&-\dot\kappa\int_{\theta,\gamma} t_0\rho\mathcal{M}_{\bar\theta}^{\kappa(\rho)\Omega}C_0\dot\kappa v(\theta)\cdot\nabla_x\rho I_1(\kappa)\psi_{\bar\theta}d\theta d\gamma\\
&+\dot\kappa\int_{\theta,\gamma} \rho\mathcal{M}_{\bar\theta}\mathcal{N}\cos(\theta-\bar\theta)\partial_t\rho\psi_{\bar\theta}d\theta d\gamma\\
&+\int_{\theta,\gamma} t_0 v(\theta)\cdot(\nabla_x\rho\mathcal{M}_{\bar\theta}\mathcal{N}\psi_{\bar\theta})d\theta d\gamma\\
&+\kappa\int_{\theta,\gamma} t_0 v(\theta)\cdot(\rho\mathcal{M}_{\bar\theta}\mathcal{N}\sin(\theta-\bar\theta)\nabla_x\bar\theta\psi_{\bar\theta})d\theta d\gamma\\
&+\dot\kappa\int_{\theta,\gamma} t_0 v(\theta)\cdot(\rho\mathcal{M}_{\bar\theta}\mathcal{N}\cos(\theta-\bar\theta)\nabla_x\rho\psi_{\bar\theta})d\theta d\gamma = 0
\end{aligned}$$

We write $X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8$ the eight terms of the equation. Using the symmetry satisfied by ψ in (5.29), the fact that ψ is odd and $\mathcal{M}(\theta)\mathcal{N}(\gamma)$ and $\mathcal{M}(\theta)\mathcal{N}(\gamma)\cos(\theta)$ are even we get $X_1 = X_3 = X_5 = X_8 = 0$. For the second term, the change of variables $\theta' = \theta - \bar\theta$ leads to

$$X_2 = \kappa\rho\partial_t\bar\theta\gamma_1 \tag{5.31}$$

with

$$\gamma_1 = \langle\sin\theta\psi\rangle_\mu \tag{5.32}$$

where the notation (5.24) has been used. For X_4 we use the change of variables $\theta' = \theta - \bar\theta$ and the symmetry satisfied by ψ to get :

$$X_4 = -\dot\kappa t_0\rho\frac{I_1(\kappa)}{I_0(\kappa)}\gamma_1\nabla_x\rho\cdot v(\bar\theta)^\perp \tag{5.33}$$

where $v(\bar\theta)^\perp = \begin{pmatrix} -\sin\bar\theta \\ \cos\bar\theta \end{pmatrix}$

The same reasoning for X_6 leads to

$$X_6 = t_0\gamma_1\nabla_x\rho \cdot v(\bar\theta)^\perp \tag{5.34}$$

The same applies reasoning for X_2 applies and for X_7 and we find :

$$X_7 = t_0\kappa\gamma_2\rho\nabla_x\bar\theta \cdot v(\theta) \tag{5.35}$$

with

$$\gamma_2 = \langle \cos\theta \sin\theta\psi\rangle_\mu \tag{5.36}$$

Combining (5.31), (5.32), (5.34), (5.35), (5.36) and the fact that $X_1 = X_3 = X_5 = X_8 = 0$ yields :

$$\kappa\rho\partial_t\bar\theta\gamma_1 + t_0\gamma_1\nabla_x\rho\cdot v(\bar\theta)^\perp + t_0\kappa\gamma_2\rho\nabla_x\bar\theta\cdot v(\theta) - \dot\kappa t_0\rho\frac{I_1(\kappa)}{I_0(\kappa)}\gamma_1\nabla_x\rho\cdot v(\bar\theta)^\perp = 0. \tag{5.37}$$

We note that $\Omega = v(\bar\theta)$ and :

$$\partial_t\Omega = \partial_t\bar\theta\Omega^\perp \quad \text{and} \quad (\Omega\cdot\nabla_x)\Omega = (\Omega^\perp\otimes\Omega)\nabla_x\bar\theta$$

Therefore multiplying equation (5.37) by $\Omega^\perp$ leads to

$$\rho\partial_t\Omega + c_2\rho(\Omega\cdot\nabla_x)\Omega + c_3(Id - \Omega\otimes\Omega)\nabla_x\rho + c_4\rho(Id-\Omega\otimes\Omega)\nabla_x\rho = 0 \tag{5.38}$$

with

$$c_2 = t_0\frac{\gamma_2}{\gamma_1} \tag{5.39}$$

$$c_3 = \frac{t_0}{\kappa} \tag{5.40}$$

$$c_4 = \frac{t_0\dot\kappa I_1(\kappa)}{\kappa I_0(\kappa)} \tag{5.41}$$

□

5.6 The order 1 in ε in the ordered region

As in the disordered region, we are interested by the first order in ε of f^ε in the ordered region. For this reason we write :

$$f^\varepsilon = \rho(x,t)\mu_{\bar\theta}^{J_{f^\varepsilon}} + \varepsilon f_1^\varepsilon \tag{5.42}$$

For sakeness of clarity, we drop the ε index on the functions. (5.42) leads to the following equation for J_f.

$$J_f = \rho c(|J_f|)J_f + \varepsilon J_{f_1} \qquad c(|J_f|) = \frac{I_1(|J_f|)}{|J_f|I_0(|J_f|)} \tag{5.43}$$

We find the following theorem :

Theorem 5.3. *When $\varepsilon \to 0$, the formal (first) order approximation to the solution of the rescaled equation (3.6) for $f^\varepsilon \in D(A)$ in the ordered region is given by (dropping ε) :*

$$f(t,x,\theta,\gamma) = \rho(x,t)\mu_{\bar\theta}^{J_f} + \varepsilon\Big(\chi_o + \frac{\varphi(\theta,t,x)\mu}{\gamma}\Big) \tag{5.44}$$

where χ_o has been defined in the proof. φ is defined as follows :

$$\varphi(\theta) = \exp(-J_f\cdot v(\theta))\Big(\Big(-\frac{t_0}{\beta_1} - \frac{\rho}{\beta_1}\frac{|\dot J_f| I_1(|J_f|)}{I_0(|J_f|)}\Big)\partial_t\rho\int^\theta \exp(J_f\cdot v(s))ds$$
$$+ A(t,x)\cdot\int^\theta v(s)\exp(J_f\cdot v(s))ds - \frac{t_0}{\beta_1}\int^\theta v(s)\cdot\nabla_x(J_f\cdot v(s))\exp(J_f\cdot v(s))ds\Big)$$

where
$A(t,x) = -\frac{1}{\beta_1}\Big(t_0\nabla_x\rho + \rho\frac{dJ_f}{dt} - \nabla_x|J_f|\frac{I_1(|J_f|)}{I_0(|J_f|)}\Big)$

Proof. We remind that $c(|J_f|) \to 1/2$ when $|J_f| \to 0$.

Writing μ instead of $\mu_{\bar\theta}^{J_f}$ for sake of clarity and using (3.6) leads to :

$$\begin{aligned}(5.45)\quad &\varepsilon(\partial_t\rho\mu + t_0 v(\theta)\cdot\nabla_x\rho\mu + \rho\mu\frac{dJ_f}{dt}\cdot v(\theta) + t_0 v(\theta)\cdot(v(\theta)\nabla_x\cdot J_f)\mu -\\ &\rho\mu\frac{\big(|\dot J_f| + v(\theta)\cdot\nabla_x|J_f|\big)I_1(|J_f|)}{I_0(|J_f|)}) + t_0 v(\theta)\cdot\nabla_x + \varepsilon^2(\partial_t f_1 + t_0 v(\theta)\cdot\nabla_x f_1)\\ &\qquad = -\varepsilon\beta_1\gamma\partial_\theta f_1 - \varepsilon\beta_1\bar\gamma^f\partial_\gamma f_1 + \varepsilon\partial_\gamma\Big(\mathcal{N}\partial_\gamma\Big(\frac{f_1}{\mathcal{N}}\Big)\Big)\end{aligned}$$

with $|\dot J_f|$ being the derivative of $|J_f|$ with respect to t. Since $c(|J_f|)$ is decreasing and bounded by $1/2$ we cannot have $(1-\rho c(|J_f|)) = 0$. Otherwise this would lead to $\rho \le 2$ which is not possible in the ordered region. Therefore $(1-\rho c(|J_f|)) \neq 0$ Multiplying (5.46) by $(1-\rho c(|J_f|))$ and neglecting the term in ε^2 we find :

$$\begin{aligned}(5.46)\quad &-\beta_1\gamma\partial_\theta f_1 + \partial_\gamma\Big(\mathcal{N}\partial_\gamma\Big(\frac{f_1}{\mathcal{N}}\Big)\Big)\\ &= \partial_t\rho\mu + t_0 v(\theta)\cdot\nabla_x\rho\mu + \rho\mu\frac{dJ_f}{dt}\cdot v(\theta) + t_0 v(\theta)\cdot(v(\theta)\nabla_x\cdot J_f)\mu - \rho\mu\frac{|\dot J_f| I_1(|J_f|)}{I_0(|J_f|)} - \rho\mu v(\theta)\cdot\nabla_x|J_f|\frac{I_1(|J_f|)}{I_0(|J_f|)}\end{aligned}$$

We decompose f_1 in the following way :

$$f_1(t,x,\theta,\gamma) = f_2(t,x,\theta,\gamma) + \frac{\varphi(\theta,t,x)\mu}{\gamma}$$

with a fixed φ satisfying :

$$\begin{aligned}\varphi'(\theta) + J_f\cdot v'(\theta)\varphi(\theta) = &-\frac{t_0}{\beta_1}\partial_t\rho - \frac{\rho}{\beta_1}\frac{|\dot J_f| I_1(|J_f|)}{I_0(|J_f|)} - v(\theta)\frac{\rho}{\beta_1}\frac{\nabla_x|J_f| I_1(|J_f|)}{I_0(|J_f|)} - \frac{t_0}{\beta_1}v(\theta)\cdot\nabla_x\rho - \frac{\rho}{\beta_1}\frac{dJ_f}{dt}\cdot v(\theta)\\ &- \frac{t_0}{\beta_1}v(\theta)\cdot\nabla_x(J_f\cdot v(\theta)) - \frac{\rho}{\beta_1}v(\theta)\cdot\nabla_x|J_f|\frac{I_1(|J_f|)}{I_0(|J_f|)}\end{aligned}$$

We define $A(t,x) = -\frac{1}{\beta_1}\Big(t_0\nabla_x\rho + \rho\frac{dJ_f}{dt} - \nabla_x|J_f|\frac{I_1(|J_f|)}{I_0(|J_f|)}\Big)$ with and we get :

$$\varphi'(\theta) + J_f \cdot v'(\theta)\varphi(\theta) = -\frac{t_0}{\beta_1}\partial_t \rho + A(t,x)\cdot v(\theta) - \frac{t_0}{\beta_1} v(\theta)\cdot \nabla_x (J_f \cdot v(\theta)) \tag{5.47}$$

Searching for a solution of (5.47) leads to

$$\begin{aligned}\varphi(\theta) = \exp(-J_f \cdot v(\theta))\Big(\big(-\frac{t_0}{\beta_1} - \frac{\rho}{\beta_1}\frac{|\dot{J}_f| I_1(|J_f|)}{I_0(|J_f|)}\big)\partial_t\rho \int^\theta \exp(J_f\cdot v(s))ds \\ + A(t,x)\cdot \int^\theta v(s)\exp(J_f \cdot v(s))ds - \frac{t_0}{\beta_1}\int^\theta v(s)\cdot\nabla_x(J_f\cdot v(s))\exp(J_f\cdot v(s))ds\Big)\end{aligned}$$

Finally it only remains to find f_2 which satisfies :

$$-\beta_1\gamma\partial_\theta f_2 + \partial_\gamma\Big(\mathcal{N}\partial_\gamma\Big(\frac{f_2}{\mathcal{N}}\Big)\Big) = \varphi(\theta)\mathcal{M}_{\bar{\theta}}\partial_\gamma\Big(\frac{\mathcal{N}}{\gamma^2}\Big) \tag{5.48}$$

Thanks to proposition 5.2 we define a solution χ_o of (5.48). This ends the proof. □

6 Stability of the equilibria in the spatially homogeneous setting

Now that we have an idea of the possible solutions we want to find how our problem converges towards the equilibrium and its stability. It means that if we take f^ε close to an equilibria the problem is to find if f^ε converges to this equilibria as $t \to \infty$ and if possible find its rate of convergence. In order to study this problem we shall distinguish two cases as previously which depend on the value of ρ compared to the threshold 2. Before studying the stability we first recall the following result which is the mass conservation equation of the system.

$$\int_{\theta,\gamma} Q(f)d\theta d\gamma = 0 \tag{6.1}$$

We remind the definition of global rate.

Definition 6.1. Let $\mathcal{X}$ be a Banach space with norm $\|.\|$ and let $f(t) : \mathbb{R}_+$ be a function of t with values in $\mathcal{X}$. We say that $f(t)$ converges to f_∞ with global rate r if and only if there exists a constant C which only depends on f_0, such that

$$\|f(t) - f_\infty\| \leq Ce^{-rt}$$

6.1 Free energy

Free energy methods are often used to study the stability of the equilibria. For this purpose we try to look for a conservation relation and we rewrite (2.11) in the space homogeneous setting (i.e $\nabla_x f = 0$) as :

$$\partial_t f = \beta_1\{H_{\bar{\theta}}, f\} + \partial_\gamma\Big(f\partial_\gamma\Big(\ln\frac{f}{\mathcal{N}}\Big)\Big)$$

where (4.4) has been used. From now on and for sake of clarity we write H_{J_f} instead of $H_{\bar{\theta}}$ with

$$H_{J_f} = -J_f \cdot v(\theta) + \frac{\gamma^2}{2} \tag{6.2}$$

Equation (2.11) together with (6.2) becomes :

$$\partial_t f = \beta_1 \left\{ H_{J_f}, f \right\} + \partial_\gamma \Big(f \partial_\gamma \Big(\ln \frac{f}{\mathcal{N}} \Big) \Big) \tag{6.3}$$

Since any solution is positive for any $t > 0$ there is no problem using $\ln f$.
Therefore :

$$\int_{\theta,\gamma} \partial_t f \big(\ln \frac{f}{\mathcal{N}} - J_f \cdot v(\theta) \big) d\theta d\gamma = \beta_1 \int_{\theta,\gamma} \left\{ H_{J_f}, f \right\} \big(\ln \frac{f}{\mathcal{N}} - J_f \cdot v(\theta) \big) d\theta d\gamma - \int_{\theta,\gamma} f \Big| \partial_\gamma \big(\ln \frac{f}{\mathcal{N}} \big) \Big|^2 d\theta d\gamma$$

For the first term of the right hand side we get after applying Green's formula :

$$\int_{\theta,\gamma} \left\{ H_{J_f}, f \right\} \big(\ln \frac{f}{\mathcal{N}} - J_f \cdot v(\theta) \big) d\theta d\gamma = 0$$

Hence the previous equation reduces to the following conservation relation :

$$\int_{\theta,\gamma} \partial_t f \Big(\ln \Big(\frac{f}{\mathcal{N}} - J_f \cdot v(\theta) \Big) \Big) d\theta d\gamma = - \int_{\theta,\gamma} f \Big| \partial_\gamma \Big(\ln \Big(\frac{f}{\mathcal{N}} \Big) \Big) \Big|^2 d\theta d\gamma \tag{6.4}$$

We define :

$$\mathcal{F}(f) = S\Big(\frac{f}{\mathcal{N}}\Big) - \rho_f - \frac{|J_f|^2}{2} \quad \text{with } S(f) = \int_{\theta,\gamma} f \ln(f) \mathcal{N} d\theta d\gamma \tag{6.5}$$

$$\mathcal{D}(f) = \int_{\theta,\gamma} f \Big| \partial_\gamma \Big(\ln \Big(\frac{f}{\mathcal{N}} \Big) \Big) \Big|^2 d\theta d\gamma \tag{6.6}$$

And (6.4) reads :

$$\frac{d}{dt} \mathcal{F} + \mathcal{D} = 0 \tag{6.7}$$

Since in the homogeneous setting, we have $\partial_t \rho = 0$ we define :

$$\widetilde{\mathcal{F}}(f) = S\Big(\frac{f}{\mathcal{N}}\Big) - \frac{|J_f|^2}{2} \tag{6.8}$$

and (6.7) applies for $\widetilde{\mathcal{F}}$ instead of $\mathcal{F}$
We find the following proposition :

Proposition 6.1 (Stationary solutions)**.** *The stationary solutions of (6.3) f satisfy the following properties :*
(i) Equilibrium : $Q(f) = 0$.
(ii) No dissipation : $\mathcal{D}(f) = 0$.
(iii) There exists C a positive constant such that $\ln(f) + H_{J_f} = C$.
(iv) The probability density f is positive and a critical point of $\widetilde{\mathcal{F}}$ and $\mathcal{F}$ (under the constraint of mean 1). Moreover (i) (iii) and (iv) are equivalent and imply (ii).

Proof. A stationary solution is a solution independant of t. We get that $Q(f) = 0$. With the conservation relation (6.7) we get that $\frac{d}{dt}\mathcal{F} = 0$, so $\mathcal{D}(f) = 0$. (iii) results from (i) and using the study of the collision operator.

For (iv) we do a variational study of $\widetilde{\mathcal{F}}$ around f. It means that we take a small perturbation $f + h$ so that f remains a probability density function (i.e $\int_{\theta,\gamma} h d\theta d\gamma = 0$).

We can expand the function $x \mapsto x \ln x$ around f since $f > 0$, and we have :

$$\widetilde{\mathcal{F}}(f+h) = \int_{\theta,\gamma} f \ln\left(\frac{f}{\mathcal{N}}\right) + h \ln\left(\frac{f}{\mathcal{N}}\right) d\theta d\gamma - \frac{|J_f|^2}{2} - J_f \cdot \int_{\theta,\gamma} h v(\theta) d\theta d\gamma + O(\|h\|_\infty^2)$$

which is (using (iii) and the fact that $\int_{\theta,\gamma} h = 0$) :

$$\begin{aligned}\widetilde{\mathcal{F}}(f+h) &= \widetilde{\mathcal{F}}(f) + \int_{\theta,\gamma} h\Big(\ln\left(\frac{f}{\mathcal{N}}\right) - J_f \cdot v(\theta)\Big) d\theta d\gamma + O(\|h\|_\infty^2) \\ &= \widetilde{\mathcal{F}}(f) + O(\|h\|_\infty^2)\end{aligned}$$

The reasoning is the same for $\mathcal{F}$. We get (iv). In order to show the equivalence of (iii) and (iv) it only remains to show (iii) starting with (iv). We suppose (iii). Then for every h under the constraint $f + h$ of mean 1 we get :

$$\int_{\theta,\gamma} h\Big(\ln\left(\frac{f}{\mathcal{N}}\right) - J_f \cdot v(\theta)\Big) d\theta d\gamma = 0$$

This means exactly (iii).

□

This proposition is different from [7] since the properties are not equivalent. Indeed $\mathcal{D}(f) = 0$ does not imply that f is an equilibrium since the derivative of f in $\mathcal{D}$ is only taken with respect to γ.

We need to show the existence of a unique minimizer of $\widetilde{\mathcal{F}}$.
For this reason we show first that $\widetilde{\mathcal{F}}$ is bounded below.

Proposition 6.2. *$\widetilde{\mathcal{F}}$ is bounded below in the set $L^1(\mathbb{R} \times [-\pi, \pi), d\theta d\gamma)$ by -1/2*

Proof. We have :

$$\widetilde{\mathcal{F}}(f) = \int_{\theta,\gamma} f \ln\left(\frac{f}{\mathcal{N}}\right) d\theta d\gamma - \frac{|J_f|^2}{2} \geq \int_{\theta,\gamma} f \ln\left(\frac{f}{\mathcal{N}}\right) d\theta d\gamma - \frac{1}{2}$$

Let $g = f/\mathcal{N}$. We shall use the fact that $g \ln g \geq g - 1$ and that $\int_{\theta,\gamma}(g-1)\mathcal{N} d\theta d\gamma = 1$ to get :

$$\widetilde{\mathcal{F}}(f) \geq -1/2$$

□

We also provide two properties of the free energy that are not used but could be helpful for future work.

Proposition 6.3. *The free energy $\widetilde{\mathcal{F}}$ has a unique nonnegative minimizer with unit mass, f_m in the set of all nonnegative probability distributions of $L^1(\mathbb{R}\times[-\pi, \pi), d\theta d\gamma)$ with mean 1 with respect to $d\theta d\gamma$:*

$$\min \widetilde{\mathcal{F}}(f) = \widetilde{\mathcal{F}}[f_m]$$

Proof. We know that $\widetilde{\mathcal{F}}$ is bounded below by $-1/2$ with the previous proposition. We shall also use the fact that for probability distributions $L^1(\mathbb{R}\times[-\pi,\pi), d\theta d\gamma)$ is equi-integrable and hence, with Dunford-Pettis theorem, every minimizing sequence is relatively compact in $L^1(\mathbb{R}\times[-\pi,\pi), d\theta d\gamma)$. We also have $f \to J_f$ is relatively compact. The existence of a minimizer f_m follows by lower-semi-continuity. $\square$

Proposition 6.4. *Let f be a solution of (6.3) with initial condition f_0 such that $\widetilde{\mathcal{F}}(f_0) \leq +\infty$. Then the free energy satisfies :*

$$\widetilde{\mathcal{F}}(f)(t) \leq e^{-\mathcal{K}_0 t}\Big(\widetilde{\mathcal{F}}(f_0) - \frac{\mathcal{K}_0}{2}\int_0^t |J_f|^2(s)e^{\mathcal{K}_0 s}ds\Big)$$

where $\mathcal{K}_0$ is the constant of the logarithmic Sobolev inequality and is defined in the proof.

Proof. We use the fact that :

$$\mathcal{D}(f) \geq \mathcal{K}_0 \int_{\theta,\gamma} f\ln(\frac{f}{\mathcal{N}}) = \mathcal{K}_0(\widetilde{\mathcal{F}}(f) + \frac{|J_f|^2}{2})$$

where the logarithmic Sobolev inequality has been used. Applying the Grönwall's inequality to $\widetilde{\mathcal{F}}$ since $\frac{d}{dt}\widetilde{\mathcal{F}} = -\mathcal{D}$ gives the result. $\square$

The following two properties show the stability of the isotropic stationary solutions.

For a stationary solution f_s of (6.3) and g such that $\int_{\theta,\gamma} g f_s d\theta d\gamma = 0$ we define

$$q_1[g] := \lim_{\varepsilon\to 0}\frac{2}{\varepsilon^2}\Big(\widetilde{\mathcal{F}}[f_s(1+\varepsilon g)] - \widetilde{\mathcal{F}}[f_s]\Big) \tag{6.9}$$

Proposition 6.5. *We have :*

$$q_1[g] = \int_{\theta,\gamma} g^2 f_s d\theta d\gamma - \frac{|w_g|^2}{2} \quad \textit{with} : w_g = \int_{\theta,\gamma} g f_s v(\theta) d\theta d\gamma$$

Proof. This results from a Taylor expansion and using the fact that the stationary solutions are critical point of the free energy. $\square$

Proposition 6.6 (Stability of isotropic stationary solution)**.** *On the space of the functions $g \in L^2(f_s d\theta d\gamma)$ such that $\int_{\theta,\gamma} g f_s d\theta d\gamma = 0$, q_1 is a positive quadratic form in the disored region (i.e $\rho < 2$). This proves the linear stability of the stationary solution f_s.*

Proof. We use proposition 6.5. Applying Cauchy-Schwarz inequalty on w_g yields :

$$|w_g|^2 \leq \int_{\theta,\gamma} g^2 f_s d\theta d\gamma\Big(\int_{\theta,\gamma} f_s\cos^2\theta d\theta d\gamma + \int_{\theta,\gamma} f_s \sin^2\theta d\theta d\gamma\Big) = \rho\int_{\theta,\gamma}\int_{\theta,\gamma} g^2 f_s d\theta d\gamma$$

This yields

$$q_1[g] \geq (1-\frac{\rho}{2})\int_{\theta,\gamma} g^2 f_s d\theta d\gamma$$

Since $\rho < 2$, q_1 is a positive quadratic form and this complete the proof. $\square$

We recall that a stationary solution is equivalent to be an equilibrium. We find that the isotropic stationary solutions are stable. The same method doesn't apply in the ordered region as ρ could be greater than the threshold 2. Moreover we do not know yet if any solutions of (6.3) solutions converges to stationary solutions (i.e equilibria of the collision operator). Finally we would like to get a rate of convergence. A way to do that is to use an adaptation of LaSalle's invariance principle. For this purpose we make the following hypothesis :

Hypothesis 2. Every solution of (6.3) belongs to $H^1_{1/\mathcal{N}}((-\pi,\pi]\times[-\infty,\infty])\cap L^\infty((-\pi,\pi]\times[-\infty,\infty])$ where $H^1_{1/\mathcal{N}}$ represents the Sobolev set associated with norm L^2 and measure $\frac{d\theta d\gamma}{\mathcal{N}}$. We also write L^∞ instead of $L^\infty((-\pi,\pi]\times[-\infty,\infty])$. Moreover $f\to f_\infty$ in $H^1_{1/\mathcal{N}}$ when $t\to\infty$ and we have $f_\infty\in H^1_{1/\mathcal{N}}$.

Theorem 6.1 (LaSalle's invariance principle)**.** *Under Hypothesis 2 : let f_0 be a probability measure. We denote by $\mathcal{F}_\infty$ the limit of $\mathcal{F}(f(t))$ as $t\to\infty$, where f is the solution of (6.3) with initial condition f_0.*
Then the set $\mathcal{E}_\infty=\{f\in H^1_{1/\mathcal{N}}$ s.t. $\mathcal{D}(f)=0$ and $\mathcal{F}(f)=\mathcal{F}_\infty\}$ is not empty.
Furthermore $f(t)$ converges to this set of null dissipation functions (in the following sense) :

$$\lim_{t\to\infty}\inf_{g\in\mathcal{E}_\infty}\|f(t)-g\|_{H^1_{1/\mathcal{N}}}=0.$$

Proof. We follow closely the proof in [7], proposition 3.2 of the article. $\mathcal{F}(f(t))$ is decreasing in time, and bounded by $-1/2$ below, so $\mathcal{F}_\infty$ is well defined.
Let (t_n) be an unbounded increasing sequence, and suppose with Hypothesis 2 that $f(t_n)\to f_\infty$ in $H^1_{1/\mathcal{N}}$ and $f_\infty\in H^1_{1/\mathcal{N}}$ with notation from the hypothesis.
We want to prove that $D(f_\infty)=0$. For this purpose we suppose that this is not the case. We write,

$$\begin{aligned}\mathcal{D}(f)&=\int_{\theta,\gamma}\frac{|\partial_\gamma f|^2}{f}+\int_{\theta,\gamma}f\gamma^2 d\theta d\gamma+2\int_{\theta,\gamma}\partial_\gamma f\gamma d\theta d\gamma\\&=\int_{\theta,\gamma}\frac{|\partial_\gamma f|^2}{f}+\int_{\theta,\gamma}f\gamma^2 d\theta d\gamma-2\rho_f\end{aligned}$$

Using that $f\in H$ for the second term and third term of the right hand side (i.e dividing and multiplying by $\sqrt{\mathcal{N}}$ in the integral and using Cauchy-Schwarz inequality) we get that if f_∞ is postive, then $\mathcal{D}$, as a function from the negative elements of $H^1_{1/\mathcal{N}}$ to $[0,\infty]$ is continuous at the point f_∞. In particular since $\mathcal{D}(f_\infty)>0$, there exists $\delta>0$ and $m>0$ such that if $\|f-f_\infty\|_{H^s}\le\delta$ then we have $D(f)\ge m$. The case for f_∞ only nonnegative is similar and can be found in [7]. Following [7] we find N large enough and τ such that for $n\ge N$, $\mathcal{D}(f)\ge m$ on $[t_n,t_n+\tau]$. Up to extracting, we can assume that $t_{n+1}\ge t_n+\tau$, so we have :

$$\mathcal{F}(f(t_N))-\mathcal{F}(f(t_{N+p}))=\int_{t_N}^{t_{N+p}}\mathcal{D}(f)\ge p\tau m.$$

Since the left terme is bounded by $\mathcal{F}(t_N)-\mathcal{F}_\infty$, taking p sufficiently large gives the contradiciton.
Now if we suppose that the distance (in $H^1_{1/\mathcal{N}}$ norm) between $f(t)$ and $\mathcal{E}_\infty$ does not tend to 0, we get $\varepsilon>0$ and a sequence t_n such that for all $g\in\mathcal{E}_\infty$, we have $\|f(t_n)-g\|_{H^1_{1/\mathcal{N}}}\le\varepsilon$. With Hypothesis 2 we have $f(t_n)$ is converging in H^s to f_∞. By the previous argument we have $D(f_\infty)=0$. Since $\mathcal{F}$ is decreasing in time we have that $\mathcal{F}(f_\infty)=\mathcal{F}_\infty$. So f_∞

belongs to $\mathcal{E}_\infty$ and then $\|f - f_\infty\|_{H^1_{1/\mathcal{N}}} \geq \varepsilon$ for all n. This is a contradiction. Since the distance between $f(t)$ and $\mathcal{E}_\infty$ tends to 0, this set is not empty.

□

It is important to note that we still do not know if the set $\mathcal{E}_\infty$ is only composed by equilibria. We just know that it is composed by some functions with zero dissipation i.e of the form $r(\theta)\mathcal{N}$ with $r(\theta)$ a function of θ. We leave for future work the fact than r must be the Fisher Von-Mises distribution. The idea would be to show that $\widetilde{\mathcal{F}}$ is still decreasing when $\mathcal{D}(f) = 0$ where f is a function with no dissipation which is not an equilibrium.

6.2 Estimates of convergence in the disordered region

We recall the following definition of the set H :

$$H = \{u : (-\pi, \pi] \times \mathbb{R}_\gamma \to \mathbb{R} / \int_{\theta,\gamma} \frac{u(\theta,\gamma)^2}{\mathcal{N}} d\theta d\gamma < +\infty\}$$

and define :

$$\|u\|_H = \sqrt{\int_{\theta,\gamma} \frac{u(\theta,\gamma)^2}{\mathcal{N}} d\theta d\gamma}$$

The disordered region has an equilibrium of the form $f = \rho\mathcal{N}(\gamma)\mathbb{1}_{\theta\in[-\pi,\pi]}/2\pi$
For this reason we write

$$f^\varepsilon = \frac{\rho_{f^\varepsilon}}{2\pi}\mathcal{N}(\gamma)\mathbb{1}_{\theta\in[-\pi,\pi]} + g \tag{6.10}$$

Notice that

$$\int_{\theta,\gamma} g d\theta d\gamma = 0$$

(3.6) takes the following form (omitting the ε and the indicator function for the sake of clarity, taking $\beta_2 = 1$ as before) :

$$\varepsilon(\partial_t(g) + \vec{v}(\theta)\cdot\nabla_x(g)) + \varepsilon\mathcal{N}(\gamma)\frac{\partial_t\rho_{f^\varepsilon}}{2\pi} = \\ -\beta_1\gamma\partial_\theta g - \beta_1\bar{\gamma}^g\partial_\gamma g + \beta_1\bar{\gamma}^g\gamma\frac{\rho_{f^\varepsilon}}{2\pi}\mathcal{N}(\gamma) + \partial_\gamma(\gamma g) + \partial_{\gamma^2} g$$

where $\bar{\gamma}^g = v(\theta) \times J_g$. Notice that $J_g = J_f$ and thus $\bar{\gamma}^g = \bar{\gamma}^f$.

In the spatially homogenous setting, we let $\nabla_x g = 0$. Now we integrate the previous equations with respect to (θ,γ). Using (6.1) we find $\partial_t\rho = 0$ which means that ρ does not depend on t in the homogeneous setting. The homogenous equation takes the form (writing $\bar{\gamma}$ instead of $\bar{\gamma}^g$) :

$$\varepsilon\partial_t g = \beta_1\big(-\gamma\partial_\theta g - \bar{\gamma}\partial_\gamma g + \bar{\gamma}\gamma\frac{\rho}{2\pi}\mathcal{N}(\gamma)\big) + \partial_\gamma(\gamma g) + \partial_{\gamma^2} g \tag{6.11}$$

We are interested by the stability of the equilibriums in the disordered region which means that $\rho < 2$. We note $Q_\rho(g)$ the collision operator associated with (6.11) so that $\varepsilon\partial_t g = Q_\rho(g)$.

If we apply the same reasoning than in proposition 4.1 we get :

$$(6.12)\quad \int_{\theta,\gamma} Q_\rho(g)\frac{g}{\mathcal{N}}d\theta d\gamma$$
$$= -\int_{\theta,\gamma}\mathcal{N}\Big|\partial_\gamma\Big(\frac{g}{\mathcal{N}}\Big)\Big|^2 d\theta d\gamma + \frac{\rho}{2\pi}\int_{\theta,\gamma}\bar{\gamma}\gamma g d\theta d\gamma - \beta_1\int_{\theta,\gamma}\frac{\gamma\partial_\theta g g}{\mathcal{N}}d\theta d\gamma - \beta_1\int_{\theta,\gamma}\frac{\bar{\gamma}\partial_\gamma g g}{\mathcal{N}}d\theta d\gamma$$

where the notations have been defined in proposition 4.1.

We are interested in the quantity :

$$\tau(t) = \frac{\varepsilon}{2}\|g\|_H^2 \tag{6.13}$$

With (6.12) and (6.13), $|J_f|\cos(\theta-\bar{\theta}) = J_f\cdot v(\theta)$ and using Green's formula :

$$\frac{d\tau}{dt}(t) = -\int_{\theta,\gamma}\mathcal{N}\Big|\partial_\gamma\Big(\frac{g}{\mathcal{N}}\Big)\Big|^2 d\theta d\gamma \ + \frac{\rho}{2\pi}\int_{\theta,\gamma}\bar{\gamma}\gamma g d\theta d\gamma - \beta_1\int_{\theta,\gamma}\frac{\bar{\gamma}\partial_\gamma g g}{\mathcal{N}}d\theta d\gamma \tag{6.14}$$

If we can find a Poincaré inequality with weights this would lead to something such as

$$\frac{d\tau}{dt}(t) \lesssim \tau(t) + u(t)$$

and we shall conclude with Gronwall's lemma. Since we are in the disordered region we expect g to converge to 0. We will use the following Poincaré's inequality for Gaussian measures :

$$\int_{\theta,\gamma}|f-\bar{f}|^2\mathcal{N}d\theta d\gamma \le C_1\int_{\theta,\gamma}|\partial_\gamma f|^2\mathcal{N}d\theta d\gamma \tag{6.15}$$

with C a positive constant and $\bar{f}$ the mean of f defined as :

$$\bar{f} = \int_{\theta,\gamma} f(\gamma)\mathcal{N}d\theta d\gamma.$$

In order to show that u is small enough so that τ goes to 0 when $t\to\infty$ we need to have some estimations on J_g. Indeed suppose that we can take $J_g = 0$ when t is large enough. Then we would have $\frac{d\tau}{dt}(t,x) = -\int_{\theta,\gamma}\mathcal{N}|\partial_\gamma(\frac{g}{\mathcal{N}})|^2 d\theta d\gamma$ and with the Poincaré inequality (6.15) we would have our result. Of course the estimations such as $J_g\approx 0$ can only be found in the region $\mathcal{R}_d$ as $|J| = 0$ is the only solution to the compatibility relation. We note that $J_g = J_f$.

Theorem 6.2. *Under Hypothesis 2 and if we suppose that $\mathcal{E}_\infty$ is reduced to the isotropic distribution in the disordered region then (which is reasonable with the stability of stationary solution theorem) : Suppose g_0 is a measure such that $\int_{\theta,\gamma}\frac{g_0^2}{\mathcal{N}}d\theta d\gamma \le +\infty$. If we note g a solution of (6.11) in the disordered region with $g(0) = g_0$ then g converges to 0 with the following rough estimation for $\|g\|_H$:*

There exists t_1 and K such that for $t\ge t_1$ $g(t)$ converges to 0 with global rate $\frac{1}{2C_1\varepsilon}$:

$$\|g\|_H \le K\|g_0\|_H \exp\Big(-\frac{t}{2C_1\varepsilon}\Big)$$

with $C_1$the Poincaré's constant.

Proof. $\mathcal{E}_\infty$ is reduced to the isotropic distribution in the disordered region i.e the functions of the form $\frac{\rho}{2\pi}\mathcal{N}$. This leads to $g \to 0$ when $t \to \infty$ in $H^1_{1/\mathcal{N}}$ with theorem 6.1.

(6.14) gives the following estimates :

$$\frac{d\tau}{dt}(t) \leq -\int_{\theta,\gamma} \mathcal{N}\Big|\partial_\gamma\Big(\frac{g}{\mathcal{N}}\Big)\Big|^2 d\theta d\gamma \ + |\Psi(t)|$$

where

$$\Psi(t) = J_g(t)\Big(\frac{\rho}{2\pi}\int_{\theta,\gamma}|\gamma g| d\theta d\gamma + \beta_1 \int_{\theta,\gamma}\frac{\bar{\gamma}\partial_\gamma g g}{\mathcal{N}} d\theta d\gamma\Big)$$

Since $J_g = \int_{\theta,\gamma} g v(\theta) d\theta d\gamma$, mutliplying and dividing by $\sqrt{\mathcal{N}}$ and applying Cauchy-Schwarz inequality we get $|J_g| \leq \|g\|_H$. The same reasoning for the second term of ψ leads to $|\Psi(t,x)| \leq \frac{\rho}{2\pi}\|g\|_H^2 + \|\partial_\gamma g\|_H \|g\|_H^2$. Applying the Poincare's inequality and using the definition of τ yields :

$$\frac{d\|g\|_H^2}{dt} \leq \frac{2}{\varepsilon}\Big(-\frac{1}{C_1} + \rho C_2 + \|\partial_\gamma g\|_{\dot{H}^1_{1/\mathcal{N}}}\Big)\|g\|_H^2$$

where $\dot{H}^1_{1/\mathcal{N}}$ is the homegeneous sobolev space with measure $\frac{d\theta d\gamma}{\mathcal{N}}$

Applying Grönwall's to $\|g\|_H^2$ leads to :

$$\|g\|_H^2 \leq \|g_0\|_H^2 \exp\Big(-\frac{2t}{\varepsilon}\big(\frac{1}{C_1} - \frac{\rho}{2\pi}\big) + \frac{2}{\varepsilon}\int_0^t \|\partial_\gamma g\|_{\dot{H}^1_{1/\mathcal{N}}}(s) ds\Big)$$

We have $\|\partial_\gamma g\|_H \to 0$ when $t \to \infty$ with LaSalle's principle. Let t_1 such that for $t \geq t_1$, $\|\partial_\gamma g\|_H(t) \leq \frac{1}{4C_1}$ Moreover in the disordered region we expect $\frac{\rho}{2\pi}$. Hence for $t \geq t_1$ we get

$$\|g\|_H^2 \leq \|g_0\|_H^2 \exp\Big(-\frac{t}{C_1\varepsilon}\Big)\exp\Big(\frac{2}{\epsilon}\int_0^{t_1}\|\partial_\gamma g\|_{\dot{H}^1_{1/\mathcal{N}}}(s) ds - \frac{t_1}{4C_1}\Big)$$

□

6.3 In the ordered region

In this subsection, we consider that we remain in the ordered region. In the spatially homogeneous setting we know that ρ is constant. To prove it is enough to write (6.3) for $g = \frac{f}{\rho}$ and integrate with respect to (θ,γ). In the ordered region i.e $\rho \gg 2$ we suppose that the limit set $\mathcal{E}_\infty$ can takes two forms : either the isotropic equilibria or the functions of the form $\rho\mu_{\kappa\Omega}$ with $\Omega \in \mathbb{S}^1$ and $\mu_{\kappa\Omega} = \mathcal{M}_{\kappa\Omega}\mathcal{N}$. $\mathcal{M}_{\kappa\Omega(t)} = C_0\exp(\kappa\Omega \cdot v(\theta))$, C_0 being the normalization of the Von-Mises. We want to describe the set $\mathcal{E}_\infty$ based on the initial datum f_0 for f. In [7] the authors find that the condition describing the set $\mathcal{E}_\infty$ is the value of J_{f_0}. In their case, if $J_{f_0} = 0$ then J_f remains equal to 0 for all t and $\mathcal{E}_\infty$ is composed by the isotropic equilibria. In the other case if $J_{f_0} \neq 0$ then $\mathcal{E}_\infty$ is composed of the Von-Mises functions. In our case, even if $J_{f_0} = 0$ we may have $J_{f(t)} \neq 0$ for some t. Indeed we find the following ODE for J_f, multiplying (6.3) by $v(\theta)$, integrating with respect to (θ,γ) and using Green's formula :

$$\frac{dJ_f}{dt} = \beta_1\int_{\theta,\gamma}\gamma f v'(\theta) d\theta d\gamma \tag{6.16}$$

For instance with $f_0 = C\gamma\mathcal{N}(\gamma)\theta$ and C being a normalization constant we have $\frac{dJ_f}{dt}(t=0) \neq 0$. For this reason, we leave the research of the condition for the form of $\mathcal{E}_\infty$ for future work and we get the following proposition :

Proposition 6.7. *(i) Suppose $\mathcal{E}_\infty$ is reduced to the isotropic distributions. Then in any H^s :*

$$\lim_{t\to\infty} \|f - \frac{\rho}{2\pi}\mathcal{N}\|_{H^s} = 0$$

(ii) Suppose $\mathcal{E}_\infty = \{\rho\mu_{\kappa\Omega}, \Omega \in \mathbb{S}^1\}$. Then in any H^s :

$$\lim_{t\to\infty} \|f - \rho\mu_{\kappa\Omega(t)}\|_{H^s} = 0$$

where $\Omega_t = \frac{J_f(t)}{|J_f(t)|}$ is the mean direction of $f(t)$.

Proof. (i) results from theorem 6.1.
For (ii) : we suppose that $\|f(t) - \rho\mu_{\kappa\Omega(t)}\|_{H^s}$ does not tend to zero and define t_n such that $\|f(t_n) - \rho\mu_{\kappa\Omega(t_n)}\|_{H^s} \geq \varepsilon > 0$. By theorem 6.1, there exists $\Omega_n \in \mathbb{S}^1$ such that $\|f(t_n) - \rho\mu_{\kappa\Omega_n}\|_{H^s} \to 0$. Up to extracting we suppose that $\Omega_n \to \Omega_\infty \in \mathbb{S}$. So $f(t_n) \to \mu_{\kappa\Omega_\infty}$ in H^s. In particular we have that $J_f(t_n) \to c(\kappa)\Omega_\infty$ and then $\Omega(t_n) \to \Omega_\infty$. Then $\mu_{\kappa\Omega_{t_n}} \to \mu_{\kappa\Omega_\infty}$, giving that $\|f(t_n) - \rho\mu_{\kappa\Omega(t_n)}\|_{H^s} \to 0$, which is a contradiction. $\square$

7 Generalization of the model to other dimensions

As in [6] the threshold is n which is the dimension of the Vicsek model we may wonder if the factor two found in our threshold came from the dimension of the problem. To introduce the problem in dimension d, here is the reasoning. We first have the following for the i^{th} agent :

$$\frac{dx_i}{dt} = cv_i$$

where c is a constant and v a unit vector (the generalization of $v(\theta)$ from dimension 2). Now we want to introduce the curvature in other dimensions. We first look at the three dimensional case. Since dv_i/dt is orthogonal to v_i and varies with the curvature, we can write $\frac{dv_i}{dt} = \Gamma_i \times v_i$ where Γ_i is the vector of curvature. We write $d\Gamma_i = \lambda(\overline{\Gamma}_i - \Gamma_i)dt + \sqrt{2\alpha}dB_t$ and $\overline{\Gamma}_i = v_i \times J_i$ with $J_i = \frac{c}{N}\sum_{|x_i - x_j| \leq R} v_j$ as in the two dimensional problem.
Now for dimension d we want Γ_i an antisymmetric matric as $\frac{dv_i}{dt} \perp v_i$. And we have (dropping the index i) :

$$\frac{d\Gamma}{dt} = \Gamma v \quad \text{where} \quad \Gamma + \Gamma^T = 0$$

Trying to find Γ a antisymmetric matrix in dimension 3 such that $\begin{pmatrix}\Gamma_1\\ \Gamma_2\\ \Gamma_3\end{pmatrix} \times \begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} = \Gamma v$ we get $\Gamma = \begin{pmatrix}0 & -\Gamma_3 & \Gamma_2\\ \Gamma_3 & 0 & -\Gamma_1\\ -\Gamma_2 & \Gamma_1 & 0\end{pmatrix}$. Now this means that for the vector $\overline{\Gamma} = v \times J$ in dimension 3 we would get an associated matrix of the form $\overline{\Gamma} = \begin{pmatrix}0 & v_2J_1 - v_1J_2 & v_3J_1 - v_1J_3\\ v_1J_2 - v_2J_1 & 0 & v_3J_2 - v_2J_3\\ v_1J_3 - v_3J_1 & v_2J_3 - v_3J_2 & 0\end{pmatrix}$ which is $\overline{\Gamma} = J \otimes v - v \otimes J$.

Let's summazize :

Dimension 3 :

$$\begin{aligned}
\frac{dx_i}{dt} &= cv_i\\
\frac{dv_i}{dt} &= \Gamma_i \times v_i\\
d\Gamma_i &= \lambda(\overline{\Gamma}_i - \Gamma_i)dt + \sqrt{2\alpha}dB_t\\
\overline{\Gamma}_i &= v_i \times J_i\\
J_i &= \frac{c}{N}\sum_{|x_i - x_j| \leq R} v_j
\end{aligned}$$

Dimension d>3 :

$$\begin{aligned}
\frac{dx_i}{dt} &= cv_i\\
\frac{dv_i}{dt} &= \Gamma_i v_i \quad \Gamma_i + \Gamma_i^T = 0\\
d\Gamma_i &= \lambda(\overline{\Gamma}_i - \Gamma_i)dt + \sqrt{2\alpha}dB_t\\
\overline{\Gamma}_i &= J_i \otimes v_i - v_i \otimes J_i\\
J_i &= \frac{c}{N}\sum_{|x_i - x_j| \leq R} v_j
\end{aligned}$$

8 A small study in dimension 3

We study the problem in dimension 3 in order to find a possible compatibility relation and a threshold. We forget all the notations of dimension 2 for this study. Our equations reads for the i^{th} individual after change of variables :

$$\frac{dx_i}{dt} = v_i$$
$$\frac{dv_i}{dt} = \Gamma_i \times v_i$$
$$d\Gamma_i = \lambda(\overline{\Gamma}_i - \Gamma_i)dt + \sqrt{2\alpha}dB_t$$
$$\overline{\Gamma}_i = v \times J_i$$
$$J_i = \frac{1}{N}\sum_{|x_i - x_j| \leq R} v_j$$

where the vectors are of dimension 3 and v is on the sphere $v \in \mathbb{S}$. This leads to the following equation for the probability measure $f(t, x, v, \Gamma)$:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \Big((\Gamma \times v)f\Big) + \lambda \nabla_\Gamma \cdot \Big((\overline{\Gamma} - \Gamma)f\Big) = \alpha^2 \Delta_\Gamma f$$

with :

$$\overline{\Gamma} = v \times J_f \qquad J_f = \int_{v,\Gamma} v f(t, x, v, \Gamma) dv d\Gamma$$

After the same hydrodynamic scaling, change of variables than in dimension 2 we get :

$$\varepsilon(\partial_t f + t_0 v \cdot \nabla_x f) = \beta_1 \Big(- \nabla_v \cdot \Big((\Gamma \times v)f\Big) - \overline{\Gamma} \cdot \nabla_\Gamma f\Big) + \beta_2 \Big(\nabla_\Gamma \cdot (\Gamma f) + \Delta_\Gamma f\Big)$$

which is

$$\varepsilon(\partial_t f + t_0 v \cdot \nabla_x f) = Q(f) \tag{8.1}$$

with

$$Q(f) = \beta_1 \Big(- \nabla_v \cdot \Big((\Gamma \times v)f\Big) - \overline{\Gamma} \cdot \nabla_\Gamma f\Big) + \beta_2 \Big(\nabla_\Gamma \cdot (\Gamma f) + \Delta_\Gamma f\Big)$$

than we can rewrite as :

$$Q(f) = \beta_1 v \cdot \Big(\Gamma \times \nabla_v f + \nabla_\Gamma f \times J_f\Big) + \beta_2 \Big(\nabla_\Gamma \cdot (\Gamma f) + \Delta_\Gamma f\Big)$$

We define :

$$H_{J_f} = -J_f \cdot v + \frac{|\Gamma|^2}{2}$$

and we get :

$$Q(f) = \beta_1 v \cdot \Big(\nabla_\Gamma H_{J_f} \times \nabla_v f - \nabla_\Gamma f \times \nabla_v H_{J_f}\Big) + \beta_2 \Big(\nabla_\Gamma \cdot (\Gamma f) + \Delta_\Gamma f\Big)$$

For clarity we define a new Bracket :

Definition 8.1. Let A and B be two functions of (v, Γ). Then :

$$\{A, B\}^{\times}_{v,\Gamma} = \nabla_\Gamma B \times \nabla_v A - \nabla_\Gamma A \times \nabla_v B$$

We also define :

$$\mathcal{N}(\Gamma) = \frac{1}{(2\pi)^{3/2}} \exp(-\frac{|\Gamma|^2}{2}) \tag{8.2}$$

With this definition we get (dropping the index of the bracket for sake of clarity) :

$$Q(f) = \beta_1 v \cdot \{f, H\}^{\times} + \beta_2 \nabla_\Gamma \cdot \Big(\mathcal{N} \nabla_\Gamma \Big(\frac{f}{\mathcal{N}}\Big)\Big)$$

In particular, the Gaussian $\mathcal{N}$ is in the kernel of the right hand side and every function of H is in the kernel of the left hand side. We combine the two previous observations to define the function :

$$\mu^J(\theta, \gamma) = C \exp\Big(- H^J\Big) = C \exp\Big(- \Big(\frac{|\Gamma|^2}{2} - J \cdot v\Big)\Big) \tag{8.3}$$

where C is the normatization constant such that the integral of μ is one.

We define for a unit vector $\Omega \in \mathbb{S}$ and $\kappa \geq 0$ the Fisher-Von Mises distribution with concentration parameter κ and orientation Ω by :

$$\mathcal{M}_{\kappa\Omega}(\omega) = \frac{\exp(\kappa\omega \cdot \Omega)}{\int_{\mathbb{S}} \exp(\kappa v \cdot \Omega) dv} \tag{8.4}$$

We can write μ as a product of $\mathcal{M}$ given by and $\mathcal{N}$:

$$\mu_J(\theta, \gamma) = \mathcal{M}_J(v) \mathcal{N}(\gamma) \tag{8.5}$$

The properties of Q are the same than for the two dimensional case. For this reason we do not write again the proof which works in the same way.

Proposition 8.1. *i) The operator Q satisfies :*

$$\int_{v,\Gamma} Q(f) \frac{f}{\mu_{J_f}} dv d\Gamma = -\beta_2 \int_{v,\Gamma} \frac{\mathcal{N}}{\mathcal{M}_{J_f}} \Big| \nabla_\Gamma \Big(\frac{f}{\mathcal{N}}\Big)\Big|^2 dv d\Gamma \leq 0 \tag{8.6}$$

ii) If f is an equilibria of Q (i.e the functions $f(v, \Gamma) \geq 0$ such that $Q(f) = 0$) then f is of the form :

$$f = \rho \mu_{J_f} \;\; \textit{with} \;\; \rho \in \mathbb{R}^+ \tag{8.7}$$

where ρ is the total mass.

Repeating the calcul from dimension 2, we find that the compatibility relation is :

$$J_f = \rho \frac{\int_{\mathbb{S}} \exp\Big(J_f \cdot v\Big) v dv}{\int_{\mathbb{S}} \exp\Big(J_f \cdot v\Big) dv} \tag{8.8}$$

And the threshold found is $\zeta = 3$.

9 Conclusion

In this work, we have modified the PTWA model by taking off the normalization of the force acting on the individuals. This modification led to equilibria implicitly given and linked to a compatibility equation. This relation showed a threshold for the density of individuals and suggested to tackle down the problem in two regions. In the disoredered region, the system converges towards a Gaussian function of curvature and uniform distribution in angle. The equilibrium is shown to be stable and with an exponential asymptotic rate, using free energy techniques along with a hypothesis and a LaSalle's invariance principle. In the ordered region, the system converges towards a Gaussian function of curvature and a Fisher Von-Mises distribution in angle. The equilibrium is shown to be stable. In the last sections, a generalization of the model in greater dimensions is suggested.
In future work, we shall study the asymptotic rate in the ordered region which may be found using inequalities on the free-energy. We also intend to analyze the well-posedness of the problem and explore the dynamics in greater dimensions.

References

[1] **C. Carere, C.K. Hemelrijk and H. Hildenbrandt**, *Self-organized aerial displays of thousands of starlings : a model*, Behavioral Ecology doi :10.1093/beheco/arq149

[2] **A. Cavagna, L. Del Castello, I. Giardina, T. Grigera, A. Jelic, S. Melillo, T. Mora, L. Parisi, E. Silvestri, M. Viale, A. M. Walczak**, *Flocking and turning : a new model for self-organized collective motion*, arXiv :1403.1202.

[3] **P. Degond, A. Frouvelle and J-G. Liu**, *Macroscopic Limits and Phase Transition in a System of Self-propelled Particles*, arXiv :1109.2404.

[4] **P. Degond and S. Motsch**, *Continuum limit of self-driven particles with orientation interaction*, arXiv :0710.0293.

[5] **P. Degond and S. Motsch**, *Large scale dynamics of the Persistent Turning Walker model of fish behavior*, arXiv :0710.4996.

[6] **P. Degond and S. Motsch**, *A macroscopic model for a system of swarming agents using curvature control*, arXiv :1010.5405.

[7] **A. Frouvelle and J-G. Liu**, *Dynamics in a kinetic model of oriented particles with phase transition*, arXiv :1101.2380.

[8] **J. Gautrais, S. Motsch, C. Jost, M. Soria, A. Campo, R. Fournier, S. Bianco and G. Théraulaz**, *Analyzing fish movement as a persistent turning walker*, J. Math. Biol. (2009) 58 :429–445.

[9] **C.K. Hemelrijk and H. Hildenbrandt**, *Schools of fish and flocks of birds : their shape and internal structure by self-organization*, Interface Focus 2012 2, doi : 10.1098/rsfs.2012.0025

[10] **C.K. Hemelrijk and H. Hildenbrandt**, *Some Causes of the Variable Shape of Flocks of Birds*, PLoS ONE 6(8) : e22479. doi :10.1371/journal.pone.0022479

[11] **C.K. Hemelrijk and H. Hildenbrandt, J. Reinders and E.J. Stamhuis**, *Emergence of Oblong School Shape : Models and Empirical Data of Fish*, doi : 10.1111/j.1439-0310.2010.01818.x

[12] **N. Jiang, L. Xiong and T. Zhang**, *Hydrodynamic limits of the kinetic self-organized models* , arXiv :1508.04640

[13] **X. Li**, *Flocking : Phase transition and asymptotic behaviour* , arXiv :1906.07517