Stability of Minkowski space for the Einstein-Vlasov system in the harmonic gauge

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May - July 2018

Abstract

Christodoulou and Klainerman have shown that Minkowski space was globally stable as a solution to the Einstein-Vlasov system in the vacuum case. The system couples Einstein's equations to a kinetic matter model. In this text, which is an internship report, we follow step by step the approach of the article written by Hans Lindblad and Martin Taylor [1] and prove the stability based on a harmonic gauge which recast the Einstein equations as a system of quasilinear wave equations for the metric coupled to a transport equation for the Vlasov particle distribution function. The length of this report comes from the fact that it explains and details some concepts briefly introduced in the original article and complete its proofs when they are not detailed.

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1 Introduction

1.1 The Einstein-Vlasov system

In the spacetime of Galileo and Newton, the simultaneity of two events is a notion independent of observers. Space and time are absolute objects. The equations are written in the Euclidian space $\mathbb{R}^3$ and lenghts and time can be measured independently with standard methods. Let's consider the gravitational potential Φ of a mass M which can be approximated by $\Phi(r) = -\frac{GM}{r}$ where $r = |x|$ in cartesian coordinates and G is the gravitational constant. The resulting force on a particle of mass m is $F = -\nabla_x \Phi = -G\frac{Mm}{r^3} r$ where ∇_x represents the gradient with respect to the coordinates x. In fact, in a more general case where we consider that there is not just one mass M but a "gravitational mass density" of sources named ρ_G, the Poisson equation gives the link between the potential Φ and ρ_g and reads

$$\Delta\Phi = -4\pi\rho_g$$

where $\Delta = \sum_{i=1,2,3} \frac{\partial^2}{(\partial x^i)^2}$. This equation gives

$$\Phi(x) = \int_{\mathbb{R}^3} \frac{\rho_G(y)}{|x-y|} dy^1 dy^2 dy^3$$

and we can deduce the force $F = -\nabla_x \Phi$. This idea to consider a density of particles appear in kinetic theory. In fact in kinetic theory, the matter, usually called a plasma is composed of particles whose size is negligible at the considered scale. Because the number of particles is so great and chaotic it is impossible to observe their individual motion. It is assumed that the state of the matter in a spacetime is represented by a one-particle distribution function. This function $f(t,x,v)$ is interpreted as the density of particles at a time t, a point x, that have a velocity v in the tangent space. Plasma waves are generated by the charge separation that arises when ions and electrons are displaced from their equilibrium positions in a plasma. If we thus consider an electrical field E who derives from a potentiel Φ (we would have the same equations with a gravitational field), we have the following equations

$$E = -\nabla_x \Phi, \qquad -\Delta\Phi = 4\pi\rho, \qquad \rho(t,x) = q\int_{\mathbb{R}^3} f(x,v,t)dv$$

where q is the charge of a particle. If f is smooth then we know that

$$t \to f(x(t), v(t), t)$$

is constant when $(x(t), v(t))$ verify the following Hamiltonian system :

$$\dot{x}(t) = v(t), \qquad \dot{v}(t) = \frac{q_\alpha}{m_\alpha} E(x(t), t)$$

The Vlasov equation is then given by

$$\frac{\partial f}{\partial t} + (v \cdot \nabla_x) f + \frac{q}{m}(E \cdot \nabla_v) f = 0 \quad \text{where} \quad v \cdot \nabla_x = \sum_i v^i \frac{\partial}{\partial x^i} \quad \text{and} \quad E \cdot \nabla_v = \sum_i E^i \frac{\partial}{\partial v^i}$$

These equations are called the Vlasov-Poisson system. They are the non-relativistic analogue to the Vlasov-Einstein system. We will study a bit this system in the section "The Vlasov-Poisson system" because some results found by Claude Bardos and Pierre Degond [2] will provide us ideas for our relativistic system.

In Special Relativity, Einstein discarded the absolute time from Newton's theory and the Minkowski's space was introduced to give a new coordinate to time : $\mathbb{R}^4$. The trajectories are thus considered in the "Spacetime" and the distance between two close events in the spacetime is given by the Minkowski's metric :

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

Guided by Galileo-Newton equivalence principle, his previous theory and the equations of motions of particles, Einstein, in many articles between 1913 and 1916, decided to replace Minkowski's space by a general Lorentzian manifold together with a metric g of signature $(-, +, +, +)$ dependent of the energy content of the manifold which gave birth to General Relativity. In his tremendous work he introduced the Einstein's equation

$$\mathbf{G}_{\mu\nu} = \frac{8\pi G}{c^4} \mathbf{T}_{\mu\nu} \quad \text{where G is the gravitational constant.}$$

Most of the concepts introduced here will be described in section 2 but it is important to understand that this equation interprets the gravitational field as a modification of the spacetime metric under the influence of the matter and the energy. More precisely we have $\mathbf{G}_{\mu\nu} = \mathbf{Ric}(g)_{\mu\nu} - \frac{1}{2}R(g)\mathbf{g}_{\mu\nu}$ where R is the trace of the Ricci's tensor with respect to the metric g. The Ricci tensor which depends on the metric can be seen as the link of the spacetime's curvature to the "flat" Euclidian space.
In General Relativity, particles trajectories are geodesics curves which verify the equation

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

as we will see in the next section. $\Gamma^i_{\alpha\beta}$ represent the Christoffel symbols. This leads to the Vlasov equation $\mathbf{X}(f) = 0$ with

$$\mathbf{X} = p^\mu \partial_{x^\mu} - p^\alpha p^\beta \Gamma^i_{\alpha\beta} \partial_{p^i}$$

where the Einstein notation has been used. Finally, we have the following system

$$Ric(g)_{\mu\nu} - \frac{1}{2} R(g) g_{\mu\nu} = T_{\mu\nu} \tag{1.1}$$

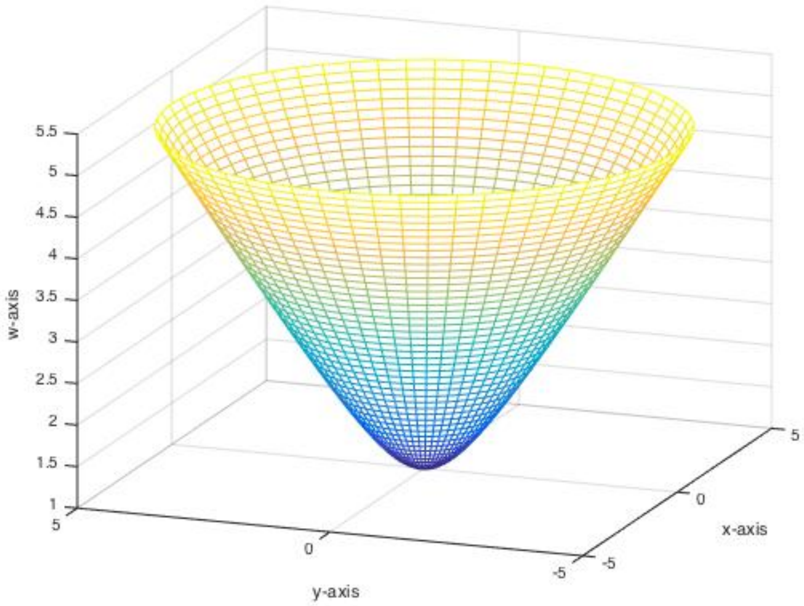
$$T^{\mu\nu}(t,x) = \int_{\mathcal{P}_{(t,x)}} f(t,x,p) p^\mu p^\nu \frac{\sqrt{|detg|}}{p^0} dp^1 dp^2 dp^3 \tag{1.2}$$

$$\mathbf{X}(f) = 0 \tag{1.3}$$

which is the Vlasov-Einstein system and where the unknown is a 3+1 dimensional manifold $\mathcal{M}$ with Lorentzian metric g together with a particle distribution function $f : \mathcal{P} \to [0, \infty)$, where the mass shell $\mathcal{P}$ is defined by

$$\mathcal{P} = \left\{ (t, x, p_0, p) \in T\mathcal{M} \,|(p^0, p) \text{ future directed}\,, g_{\mu\nu} p^\mu p^\nu = -m^2 c^2 \right\}$$

Here we will assume that the particles have a mass $m = 1$ and the speed of light is normalized to $c = 1$. The assumptions imply $-m^2 c^2 = -1$



The figure above shows the hyperbolic space $\mathbb{H}^3$ that can be defined as the above sheet of the two sheeted hyperboloid. It represents all the points $(x, y, z, w) \in \mathbb{R}^4$ such that $x^2 + y^2 + z^2 - w^2 = 1$ The z dimension has been taken off for the drawing but $\mathbb{H}^3$ should be interpreted as a volume that gives an idea of the massshell.

($\mathcal{M}$,g) is called the spacetime. Here $Rig(g)$ and $R(g)$ denote the Ricci and scalar curvature of g respectively. A coordinate system (t, x) for $\mathcal{M}$ with t a time function defines a coordinate system (t, x, p^0, p) for the tangent bundle $T\mathcal{M}$ where (t, x^i, p^0, p^i) denotes the point

$$p^0\partial_t|_{(t,x)} + p^i\partial_{x^i}|_{(t,x)} \in T\mathcal{M}$$

The Greek indices run over 0,1,2,3, lower case latin indices run over 1,2,3 and the notation $t = x^0$ is used. We will always use Einstein's notation in this report. Since the field equations under consideration are geometric in nature, we need to fix the degrees of gauge freedom before considering the stability issue from the perspective of the initial value problem. Our analysis relies on a choice of coordinate (x^μ) satisfying the homogeneous linear equation in the unknown metric. This means we impose the wave gauge condition (or harmonic gauge)

$$\Box_g x^\mu = 0 \tag{1.4}$$

where $\Box_g$ denotes the wave operator $\Box_g = g^{\alpha\beta}\partial_\alpha\partial_\beta + g^{\alpha\beta}\Gamma^\nu_{\alpha\beta}\partial_\nu$. The harmonic gauge is equivalent to the metric in the coordinates x^μ satisfying

$$g^{\alpha\beta}\partial_\alpha g_{\beta\mu} = \frac{1}{2}g^{\alpha\beta}\partial_\mu g_{\alpha\beta} \tag{1.5}$$

1.2 Global existence and stability theorem

The initial data in the Cauchy problem for the Einstein-Vlasov system consists of a 3-dimensional manifold Σ together with a metric $\overline{g}$, a symmetric (0,2) tensor k on Σ and a non-negative scalar function f_0 on the tangent bundle $T\mathcal{M}$. The relationship between the initial data ($\overline{g}$,k) and g is that there exists an embedding $i : \Sigma \to \mathcal{M}$ under the pullback of which the induced first and second fundamental form of g are $\bar{g}$ and k respectively. For the relation of the distribution functions f and f_0 we have to note that f is defined on the mass shell. The initial condition imposed is that the restriction of f to the part of the mass shell over $i(\Sigma)$ should be equal to f_0. The initial data set must satisfy the constraint equations,

$$\nabla_i k^i_j - \nabla_j(trk) = T_{0j} \qquad R + (trk)^2 - k_{ij}k^{ij} = 2T_{00}$$

A theorem of Choquet-Bruhat, guarantees that, for any such initial data set as above, there exists a globally hyperbolic solution $(\mathcal{M}, g, f)$ of the system :

Theorem. Let Σ be a 3-dimensional manifold, $\overline{g}$ a smooth Riemannian metric on Σ, k a smooth symmetric tensor on Σ and f_0 a smooth non-negative function of compact support on the tangent bundle. Suppose that these objects satisfy the constraint equations. Then there exists a smooth spacetime ($\mathcal{M}$,g), a smooth distribution function f on the mass shell of this spacetime, and a smooth embedding i of Σ into $\mathcal{M}$, which induces the given initial data on Σ such that g and f satisfy the Einstein-Vlasov system and $i(\Sigma)$ is a Cauchy surface. Moreover, given any other spacetime $(\mathcal{M}',g')$, distribution function f' and embedding i' satisfying these conditions, there exists a diffeomorphism χ from an open neighborhood of $i(\Sigma)$ in $\mathcal{M}$ to an open neighborhood of $i'(\Sigma)$ $\mathcal{M}'$ which satisfies $\chi \circ i = i'$ and carries g and g to g' and f' respectively.

Minkowski space is the trivial solution of the Einstein-Vlasov system with initial data

$$(\mathbb{R}^3, e, k \equiv 0, f_0 \equiv 0) \tag{1.6}$$

where e is the euclidian metric. The main result of this work will be to show the existence of a causally geodesically space-time asymptotically "converging" to the Minkowski space-time for an arbitrary set of smooth asymptotically flat initial data. The initial data are assumed to be asymptotically flat in the sense that Σ is diffeomorphic to $\mathbb{R}^3$ and there exists a global coordinate chart (x^1, x^2, x^3) of Σ, $M \geq 0$ which represents the "Mass" and $0 < \gamma < 1$ such that

$$\overline{g_{ij}} = (1 + \frac{M}{r})\delta_{ij} + o(r^{-1-\gamma}) \qquad k_{ij} = o(r^{-2-\gamma}), \qquad \text{as } r = |x| \to \infty$$

which comes from the positive mass theorem. We shall write

$$\overline{g_{ij}} = \delta_{ij} + \overline{h_{ij}}^0 + \overline{h_{ij}}^1, \qquad \text{where} \qquad \overline{h_{ij}}^0(x) = \chi(r)\delta_{ij}\frac{M}{r}$$

$\overline{h_{ij}}^1$ can then be deduced by substraction, χ is a smooth cut off function such that $0 \leq \chi \leq 1, \chi(s) = 1$ for $s \geq 3/4$ and $\chi(s) = 0$ for $s \leq 1/2$. We then define the initial energy,

$$\mathcal{E}_N = \sum_{|I| \leq N} (\|(1+r)^{1/2+\gamma+|I|}\nabla\nabla^I \overline{h}^1\|_{L^2(\Sigma)} + (\|(1+r)^{1/2+\gamma+|I|}\nabla^I k\|_{L^2(\Sigma)}), \tag{1.7}$$

where ∇ denotes the gradient with respect to the coordinate system (x^1, x^2, x^3). We also define,

$$\mathbb{D}_K = \sum_{k+l \leq K} \|\partial_x^k \partial_p^l f_0\|_{L^\infty} \qquad \mathcal{V}_K = \sum_{k+l \leq K} \|\partial_x^k \partial_p^l f_0\|_{L^2_x L^2_p}$$

By Sobovel inequalities there exists a constant C such that $\mathbb{D}_{K-4} \leq C\mathcal{V}_K$ for any $K \geq 0$. Indeed, we have $H^s \hookrightarrow L^\infty$ for $s > \frac{d}{2}$ where d is the dimension. $d = 7$ here and the result follows. $\mathbb{D}_K$ and $\mathcal{V}_K$ will be used to control the components of the stress-energy tensor (see Theorem 3.2).

The main result of this work is the following :

Theorem 1.1. *Let (Σ, g, k, f_0) be an initial data set for the Einstein-Vlasov system (1.1)-(1.3), asymptotically flat in the above sense, such that f_0 is compactly supported. For any $0 < \gamma < 1$ and $N \geq 11$, there exists ϵ_0 such that for all $\epsilon \leq \epsilon_0$ and all initial data satisfying*

$$\mathcal{E}_N + \mathcal{V}_N + M \leq \epsilon$$

there exists a unique future geodesically complete solution of (1.1)-(1.3), attaining the given data, together with a global system of harmonic coordinates (t, x^1, x^2, x^3) relative to which the solution asymptotically decays to Minkowski space.

The precise sense in which the spacetime of Theorem 1.1 asymptotically decay to Minkowski space will be described later. To understand this system, this theorem and the proof behind it we will first give rudiments of general relativity. The last part is the core of this report and will be dedicated to the proof of the global stability of the Minkowski space in the harmonic gauge.

Acknowledgements

I would like to convey my gratefulness to my supervisor Clément Mouhot and the Kinetic Theory Group of the Center for Mathematical Science. This internship was an amazing opportunity to acquire valuable skills and knowledge for my future carreer.
I would especially like to thank Clément Mouhot for making me work on a strikingly beautiful theme and helpful discussions. The completion of this report could not have been possible without the help of Ivan Moyano who provided me new ways of understanding and was always there when I needed some advice. My thanks also to Corentin Le Bihan from E.N.S Lyon with whom I had the opportunity to talk everyday about my report and mathematics in general. I would also like to thank Arti Sheth Thorne and the Kinetic Theory Group for their generous support and companionship, especially the following members : Josephine Evans, Megan Griffin-Pickering, Ning Jiang, Lisa Kreusser, Angeliki Menegaki, H. Xiao. Finally I would like to thank my friend Mehdi Gouyette for his support.

2 Lorentzian geometry

This section is an introduction to the mathematical concept behind general relativity.

2.1 Spacetime

In the spacetime of Galileo and Newton, the simultaneity of two events is a notion independent of observers. Space and time are absolute objects. The relativity merged space and time and used four numbers to describe an event, the time and three space coordinates.

2.1.1 Manifolds

The arena of differential geometry is a differentialble manifold. A manifold is something which is locally the same than $\mathbb{R}^n$ (in this case $n = 4$). More precisely a 4 dimension manifold is a topologic space so that on every point we can define a neighborhood homeomorphic to an open set of $\mathbb{R}^4$. A chart on a set $\mathcal{E}$ is a pair (U, ϕ) with U a subset of $\mathcal{E}$ called the domain of the chart, and ϕ a bijection from U onto an open set $\phi(U)$ of $\mathbb{R}^4$. We call coordinate system (or map) the numbers (x^0, x^1, x^2, x^3) that appear in

$$\begin{aligned} \phi : U \subset \mathcal{E} &\to \phi(U) \subset \mathbb{R}^4 \\ P &\mapsto (x^0, x^1, x^2, x^3) \end{aligned}$$

An atlas on $\mathcal{E}$ is a collection of charts (U_i, ϕ_i), $1 \leq i \leq N$ such that

$$\bigcup_{k=1}^{N} U_i = \mathcal{E}$$

We say that $\mathcal{E}$ is a differentiable manifold (resp. C^p manifold) if and only if for every non empty intersection between two maps U_i and U_j the application

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \in \mathbb{R}^4 \to \phi_i(U_i \cap U_j) \in \mathbb{R}^4$$

is differentiable (resp. C^p).

2.1.2 Vector on a manifold

A tangent vector v to a differentiable manifold $\mathcal{E}$ at a point $P \in \mathcal{E}$ is a geometric object. It can be defined through local coordinates as an equivalence class of triplets $(U_I, \phi_I, v_{\phi_I})$, where (U_I, ϕ_I) are charts containing x, while $v_{\phi_I} = (v^i_{\phi_I})$, $k = 0, ..., 3$ are vectors in $\mathbb{R}^4$. The equivalence relation is given by

$$v^i_{\phi_I} = v^j_{\phi_J} \frac{\partial^i_{x_I}}{\partial^J_{x_j}}$$

where x^i_I and x^j_J are respectively local coordinates in the charts (U_I, ϕ_I) and (U_J, ϕ_J).

To understand this concept let's take a differentiable scalar field f defined on the plane $f : \mathbb{R}^2 \to \mathbb{R}$ and let $\mathcal{C}$ be a curve in $\mathbb{R}^2$ parameterized with respect to λ $(x = X(\lambda), y = Y(\lambda))$. We have the following formula :

$$df|_{\mathcal{C}} = \frac{\partial f}{\partial x} \dot{X} d\lambda + \frac{\partial f}{\partial y} \dot{Y} d\lambda$$

Thus

$$\frac{df}{d\lambda}|_{\mathcal{C}} = \frac{\partial f}{\partial x}\dot{X} + \frac{\partial f}{\partial y}\dot{Y}$$
$$= \vec{v}\cdot\vec{\nabla}f$$

If we have the coordinate system (x^0, x^1, x^2, x^3) in $\mathcal{E}$ we define then

$$\vec{v}(f) = \sum_{\alpha=0}^{3}\frac{\partial f}{\partial x^\alpha}\frac{dX^\alpha}{d\lambda} = \frac{\partial f}{\partial x^\alpha}\frac{dX^\alpha}{d\lambda}$$

where we used the Einstein's notation for the sums. We shall use this notation from now on. We write then

$$\vec{v}(f) = \dot{X}^\alpha\vec{\partial}_\alpha(f)$$

and therefore

$$\vec{v} = \dot{X}^\alpha\vec{\partial}_\alpha$$

Definition 2.1. From this equation we deduce that the tangent vectors at a point P which are all the tangent vectors at every curves that go through P form a vector space, the tangent space to $\mathcal{E}$ of dimension 4 which is denoted $T_P(\mathcal{E})$.

Definition 2.2. The set of pairs (P, v_P), $P \in \mathcal{E}$, $v_P \in T_P(\mathcal{E})$ denoted $T(\mathcal{E})$ is called the tangent bundle to $\mathcal{E}$

2.1.3 Multilinear forms and Tensors

The linear forms on $T_P(\mathcal{E})$ build a 4 dimension vector space $T_P(\mathcal{E})^*$. We call tensor k times contravariant and l times covariant (type (k,l)) at the point P the mapping :

$$T : \underbrace{T_P(\mathcal{E})^* \times \cdots \times T_P(\mathcal{E})^*}_{k} \times \underbrace{T_P(\mathcal{E}) \times \cdots \times T_P(\mathcal{E})}_{l} \to \mathbb{R}$$
$$(\omega_1, ..., \omega_k, \vec{v}_1, ..., \vec{v}_l) \mapsto T(\omega_1, ..., \omega_k, \vec{v}_1, ..., \vec{v}_l)$$

which is linear with respect to all his components.
For instance a linear form is a $(0,1)$ tensor, a bilinear form is a $(0,2)$ tensor. Since $T_P(\mathcal{E})^{**} = T_P(\mathcal{E})$ a vector can be considered as a $(1,0)$ tensor. Tensor fields are assignments of a tensor at P to each point P of the manifold $\mathcal{E}$. For convention purposes a scalar field is a tensor field (0,0).

2.2 Metric tensor

A pseudo-Riemannian metric on a manifold $\mathcal{E}$ is a symmetric covariant 2-tensor field g such that the quadratic form it defines given in local charts by $g_{ij}X^iX^j$, is non-degenerate ; that is, the determinant does not vanish in any chart. A pseudo-Riemannian metric g on $\mathcal{E}$ defines at each point of the manifold a scalar product, that is, a bilinear non-degenerate function on the tangent space $T_P(\mathcal{E})$:

$$g(\vec{u},\vec{v}) := g_{ij}(P)u^iv^j, \qquad u, v \in T_P(\mathcal{E})$$

A metric is called Riemannian if the quadratic form defined by g is positive-definite.

A pseudo-Riemannian metric g is called a Lorentzian metric if the signature of the quadratic form defined by g is (-,+,+,+) which means that there exist a basis of $T_P(\mathcal{E})$ such that :

$$g(\vec{u},\vec{v}) = -u^0v^0 + u^1v^1 + u^2v^2 + u^3v^3$$

The bilinear form g is called the metric tensor on $\mathcal{E}$ or just metric. Moreover $(\mathcal{E}, g)$ is called spacetime.

Since g is a bilinear symetric form, $(g_{\alpha\beta})$ is a symetric matrix. Moreover, since g non-degenerate, the matrix is inversible and we note $g^{\alpha\beta}$ the inverse matrix.

$$g^{\alpha\sigma}g_{\sigma\beta} = \delta^\alpha_\beta$$

Two vectors on the tangent space are said to be orthogonal if their scalar product with the metric g vanishes.

Let $(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ be an orthonormal basis of $T_P(\mathcal{E})$ for the scalar product with respect to the metric g, which means :

$$
\begin{aligned}
\vec{e}_0 \cdot \vec{e}_0 &= -1 \\
\vec{e}_i \cdot \vec{e}_i &= 1 \qquad \text{for} \quad 1 \leq i \leq 3 \\
\vec{e}_i \cdot \vec{e}_j &= 0 \qquad \text{for} \quad i \neq j
\end{aligned}
$$

In this basis we have $g_{\alpha\beta} = g(\vec{e}_\alpha, \vec{e}_\beta) = \eta_{\alpha\beta}$ where

$$
\eta := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

is called Minkowski's matrix.

In the euclidian space of dimension 3 the signature of the scalar product is $(+,+,+)$ which provides the definite positive property. In the 4 dimensional case with a $(-,+,+,+)$ signature the metric can lose this property. At each point P of a Lorentzian manifold $\mathcal{E}$, one defines in the tangent space $T_P(\mathcal{E})$ a double cone C_P called the causal cone by the inequality

$$
g_P(\vec{u}, \vec{u}) \leq 0, \qquad \vec{u} \in T_P(\mathcal{E})
$$

The boundary of C_P in the tangent space is the double cone Γ_P, called the null cone or light cone,

$$
g_P(\vec{u}, \vec{u}) = 0
$$

A vector $\vec{u} \in T_P(\mathcal{E})$ is called causal if

$$
g_P(\vec{u}, \vec{u}) \leq 0
$$

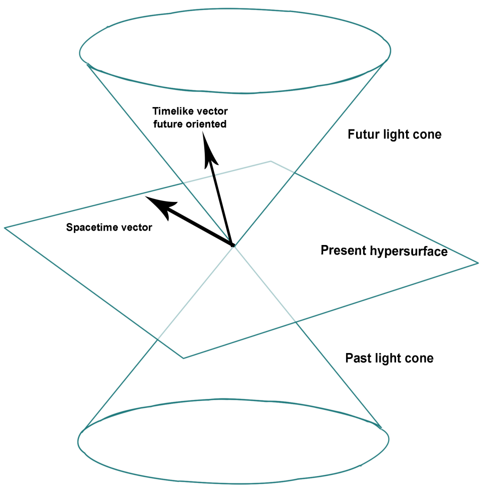
The vector is called :

— Timelike if $g_P(\vec{u}, \vec{u}) < 0$

— Spacelike if $g_P(\vec{u}, \vec{u}) > 0$

— Null if $g_P(\vec{u}, \vec{u}) = 0$

Moreover, we can split the causal cone in two convex cones : the futur light cone and the past light cone. Vectors in the futur light cone will be called futur oriented vectors and vectors in the light cone past oriented vectors.



2.3 Lenght and geodesics

Consider two points P and P' really close to each other on a manifold $\mathcal{E}$ we can associate to them a vector $\vec{dP}$ on the tangent space $T_P(\mathcal{E})$ and define the square of their distance by

$$ds^2 = g(\vec{dP}, \vec{dP})$$

If we give a coordinate system (x^α) in the neighborhood of P then we have

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$$

Postulates of the general relativity precise that the curves of the photons are null which means that every tangent vectors at every point of the curve associated to the photon are null. For a particle, in the spacetime, we postulate that the curves are timelike. Thus it is impossible for a particle to traval faster than the light.

Consider two points P and P' on infinitely close to each other with P' in the future of P. We can associate to them a vector $\vec{dP}$ and since $\vec{dP}$ is considered timelike as we previously said, we then define :

$$d\tau := \frac{1}{c}\sqrt{-g(\vec{dP}, \vec{dP})}$$

which is called the proper time of the curve. (Note the presence of the minus sign since $\vec{dP}$ is timelike)

A geodesic is a curve that maximize or minimize the distance, a geodesic is :

— Spacelike if it minimizes the distance among every spacelike curves between two points

— Timelike if it maximizes the distance among every timelike curves between two points

— Null if it is a null curve that satisfy an equation we will give below

To determine the evolution of a particle in a relativistic gravitational field we are looking for the geodesic curve between two events A and B. Consider a parameterization of the curve of the particle with respect to λ we can write

$$d\tau := \frac{1}{c}\sqrt{-g(\vec{v}, \vec{v})}d\lambda$$

If we note (x^α) a coordinate system of $\mathcal{E}$ in the neighborhood of the curve and $x^\alpha = X^\alpha(\lambda)$. We have

$$\tau(A, B) = \frac{1}{c}\int_{\lambda_A}^{\lambda_B} \sqrt{-g_{\alpha\beta}\dot{X}^\alpha \dot{X}^\beta}d\lambda$$

where $\dot{X}^\alpha = \frac{dX^\alpha}{d\lambda}$. If we call $L := \sqrt{-g_{\alpha\beta}\dot{X}^\alpha \dot{X}^\beta}$ the lagrangian that appears in the integral we have according to the Euler-Lagrange equations :

$$\frac{d}{d\lambda}\Big(\frac{\partial L}{\partial \dot{X}^\alpha}\Big) - \frac{\partial L}{\partial X^\alpha} = 0$$

This equations reads explicitly

$$\ddot{X}^\alpha + \Gamma^\alpha_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = \frac{1}{L}\frac{dL}{d\lambda}\dot{X}^\alpha$$

where we have used the symmetry in $g_{\mu\nu}$ along with the definition of the Chrisfoffel symbols :

$$R^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$

If we use τ instead of λ we have

$$\frac{d^2X^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} = 0$$

We can always found a parameter that satisfy the geodesic equation :

$$\ddot{X}^\alpha + \Gamma^\alpha_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = 0.$$

2.4 Einstein equation

In this subsection we give the Einstein's equation that we shall use but we do not describe where do these equations come from.
The Ricci tensor is :

$$Ric_{\alpha\beta} = \frac{\partial \Gamma^{\mu}_{\alpha\beta}}{\partial x^{\mu}} - \frac{\partial \Gamma^{\mu}_{\alpha\mu}}{\partial x^{\beta}} + \Gamma^{\mu}_{\alpha\beta}\Gamma^{\nu}_{\mu\nu} - \Gamma^{\nu}_{\alpha\mu}\Gamma^{\mu}_{\nu\beta}$$

The scalar curvature (also called Ricci's scallar) is :

$$R = g^{\alpha\beta} R_{\alpha\beta}$$

And we denote G the Einstein's tensor with the formula :

$$G = Ric(g) - \frac{1}{2} R g$$

The Einstein's equation give the link between the Einstein's tensor and the stress energy tensor T with

$$\mathbf{G} + \Lambda \mathbf{g} = \frac{8\pi G}{c^4} \mathbf{T}$$

where Λ is called the cosmological constant and G is the Newton's constant. Sometimes, Λ is taken equal to zero.

3 Introduction to the proof

We have now introduced different tools that will allow us to understand the proof of the stability of Minkowski space. In this section we will first recall the system and explain the Vlasov equation. Then we will present the reduced Einstein-Vlasov sytem and theorems that we need to prove Theorem 1.1. Their proof will be shown in other sections. Finally we give a parametrisation of the mass shell that will be used across the entire report.

3.1 The Vlasov equation

We consider a collision-less gas only interacting by gravity. Recall that this system takes the form

$$Ric(g)_{\mu\nu} - \frac{1}{2} R(g) g_{\mu\nu} = T_{\mu\nu} \tag{3.1}$$

$$T^{\mu\nu}(t,x) = \int_{\mathcal{P}_{(t,x)}} f(t,x,p) p^{\mu} p^{\nu} \frac{\sqrt{|detg|}}{p^0} dp^1 dp^2 dp^3 \tag{3.2}$$

$$\mathbf{X}(f) = 0 \tag{3.3}$$

where

$$\mathbf{X} = p^{\mu} \partial_{x^{\mu}} - p^{\alpha} p^{\beta} \Gamma^{i}_{\alpha\beta} \partial_{p^i}$$

Recall that the distribution function f, which represents the density of particles is a non-negative real-valued function on $\mathcal{P}$ and that in general relativity a free particle travels along a geodesic. If we consider a future-directed timelike geodesic parametrized by proper time then its tangent vector at any time is future-pointing unit timelike. Thus this geodesic has a natural lift to a curve on $\mathcal{P}$, by taking its position and tangent vector together. The condition that f represents the distribution of a collection of colisionless particles interacting only by gravity and moving freely in the given spacetime is that it should be constant along the flow on $\mathcal{P}$. This equation is the Vlasov equation. It follows in local coordinates (x, p) :

$$\frac{dx^a}{ds} = p^a \qquad \frac{dp^m}{ds} = -\Gamma^m_{bc} p^b p^c$$

Along a geodesic, the distribution function is invariant so

$$\frac{d}{ds} f(x^{\alpha}(s), p^m(s)) = 0$$

From the chain rule we get

$$p^{\mu} \partial_{x^{\mu}} f - p^{\alpha} p^{\beta} \Gamma^{i}_{\alpha\beta} \partial_{p^i} f = 0$$

which is the Vlasov equation.

3.2 The reduced Einstein-Vlasov system

Under the harmonic gauge, it has been shown that the Einstein-Vlasov system takes the form :

$$\widetilde{\Box}_g g_{\mu\nu} = F_{\mu\nu}(g)(\partial g, \partial g) + \widehat{T}_{\mu\nu} \qquad \text{where} \qquad \widetilde{\Box}_g = g^{\alpha\beta}\partial_\alpha\partial_\beta \tag{3.4}$$

and $\widehat{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}tr_g T$, $\quad tr_g T = g^{ij}T_{ij}$ and $F_{\mu\nu}(u)(v,v)$ depends quadratically on v. We note that the reduced system (3.2), (3.3) and (3.4) is a system of qualinear wave equations with a transport equation. To understand some difficulties of the system we need to recall that a typical system of quasilinear equations allows solutions with smooth arbitrarily small initial data that blow up in finite time. A condition to the global existence for such equations was found by Klainerman called the null condition. Nevertheless, Einstein's equations in the vacuum do not satisfy the null condition.

To give a better understanding of this system we shall prove the reduction of this system (see [7] lemma 3.1) in the vacuum case (i.e $T = 0$ and then $R_{\mu\nu} = 0$ where R is the Ricci tensor).

Proposition 3.1. *If g is a metric satisfying Einstein vacuum equations together with the wave coordinate condition then $g_{\mu\nu}$ solves the reduced Einstein equation*

Proof. First recall that the harmonic gauge is equivalent to the metric in the coordinates x^μ satisfying

$$g^{\alpha\beta}\partial_\alpha g_{\beta\mu} = \frac{1}{2}g^{\alpha\beta}\partial_\mu g_{\alpha\beta} \tag{3.5}$$

which is straightforward. g^{ij} is defined as the inverse of the metric tensor i.e :

$$g^{\alpha i}g_{i\beta} = \delta^\alpha_\beta \tag{3.6}$$

$$R^\lambda_{\mu\nu\delta} = \partial_\delta\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\delta} + \Gamma^\lambda_{\rho\delta}\Gamma^\rho_{\mu\nu} - \Gamma^\lambda_{\rho\nu}\Gamma^\rho_{\mu\delta} \tag{3.7}$$

is the Riemann curvature tensor, and $R_{\mu\nu} = R^\lambda_{\mu\nu\delta}$ is the Ricci tensor. The Christoffel symbols are defined as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \tag{3.8}$$

We note $R_{\alpha\beta\mu} = g_{\beta\lambda}\Gamma^\lambda_{\alpha\mu}$ and notice that

$$R_{\alpha\beta\mu} + R_{\alpha\mu\beta} = \partial_\alpha g_{\beta\mu} \qquad \Gamma_{\alpha\lambda\beta} = \Gamma_{\beta\lambda\alpha}$$

It follows that

$$g_{\alpha\lambda}R^\lambda_{\mu\nu\beta} = \partial_\beta\Gamma_{\mu\alpha\nu} - \partial_\nu\Gamma_{\mu\alpha\beta} + \Gamma_{\nu\lambda\alpha}\Gamma^\lambda_{\mu\beta} - \Gamma_{\alpha\lambda\beta}\Gamma^\lambda_{\mu\nu} \tag{3.9}$$

and the harmonic gauge allows us to show that :

$$g^{\alpha\beta}(\partial_\mu\partial_\alpha g_{\beta\nu} - \frac{1}{2}\partial_\mu\partial_\nu g_{\alpha\beta}) = -\partial_\mu g^{\alpha\beta}(\partial_\alpha g_{\beta\nu} - \frac{1}{2}\partial_\nu g_{\alpha\beta}) = g^{\alpha\alpha'}g^{\beta\beta'}\partial_\mu g_{\alpha'\beta'}(\partial_\alpha g_{\beta\nu} - \frac{1}{2}\partial_\nu g_{\alpha\beta}) \tag{3.10}$$

Hence we can write

(3.11)
$$g^{\alpha\beta}(\partial_\alpha\Gamma_{\mu\beta\nu} - \partial_\nu\Gamma_{\mu\beta\alpha}) = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}(\partial_\mu g_{\alpha'\beta'}\partial_\alpha g_{\beta\nu} + \partial_\nu g_{\alpha'\beta'}\partial_\alpha g_{\beta\mu} - \partial_\nu g_{\alpha'\beta'}\partial_\mu g_{\alpha\beta})$$

and with (3.5) one can show that

$$\begin{aligned} g^{\alpha\alpha'}g^{\beta\beta'}\partial_\mu g_{\alpha'\beta'}\partial_\alpha g_{\beta\nu} = \frac{1}{4}g^{\alpha\alpha'}g^{\beta\beta'}\partial_\mu g_{\alpha'\alpha}\partial_\nu g_{\beta\beta'} + g^{\alpha\alpha'}g^{\beta\beta'}(\frac{1}{2}(\partial_{\beta'}g_{\alpha'\alpha}\partial_\nu g_{\beta\nu} - \partial_\mu g_{\alpha'\alpha}\partial_{\beta'}g_{\beta\nu}) + \\ (\partial_\mu g_{\alpha'\beta'}\partial_\alpha g_{\beta\nu} - \partial_\alpha g_{\alpha'\beta'} - \partial_\mu g_{\beta\nu})) \end{aligned} \tag{3.12}$$

Thus

(3.13)
$$\begin{aligned} \frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}(\partial_\mu g_{\alpha'\beta'}\partial_\alpha g_{\beta\nu} + \partial_\nu g_{\alpha'\beta'}\partial_\alpha g_{\beta\mu} - \partial_\nu g_{\alpha'\beta'}\partial_\mu g_{\alpha\beta}) = g^{\alpha\alpha'}g^{\beta\beta'}(\frac{1}{4}\partial_\mu g_{\alpha'\alpha}\partial_\nu g_{\beta\beta'} - \frac{1}{2}\partial_\nu g_{\alpha'\beta'}\partial_\mu g_{\alpha\beta}) \\ + \frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}((\partial_\mu g_{\alpha'\beta'}\partial_\alpha g_{\beta\nu} - \partial_\alpha g_{\alpha'\beta'}\partial_\mu g_{\beta\nu}) + (\partial_\nu g_{\alpha'\beta'}\partial_\alpha g_{\beta\mu} - \partial_\alpha g_{\alpha'\beta'}\partial_{\beta'}g_{\beta\mu})) \\ + \frac{1}{4}g^{\alpha\alpha'}g^{\beta\beta'}((\partial_{\beta'}g_{\alpha'\alpha}\partial_\mu g_{\beta\nu} - \partial_\mu g_{\alpha'\alpha}\partial_{\beta'}g_{\beta\nu}) + (\partial_{\beta'}g_{\alpha'\alpha}\partial_\nu g_{\beta\mu} - \partial_\nu g_{\alpha'\alpha}\partial_{\beta'}g_{\beta\mu})) \end{aligned}$$

One can also show that using (3.5)

$$\Gamma_{\nu\alpha\beta}\Gamma^{\alpha\beta}_{\mu} = g^{\alpha\alpha'}g^{\beta\beta'}(\frac{1}{4}\partial_\nu g_{\alpha\beta}\partial_\mu g_{\alpha'\beta'} - \frac{1}{8}\partial_\mu g_{\beta\beta'}\partial_\nu g_{\alpha\alpha'} + \frac{1}{2}\partial_\alpha g_{\beta\mu}\partial_{\alpha'} g_{\beta'\nu}) - \frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}(\partial_\alpha g_{\beta\mu}\partial_{\beta'} g_{\alpha'\nu} - \partial_{\beta'} g_{\beta\mu} - \partial_\alpha g_{\alpha'\nu}) \tag{3.14}$$

Taking the trace of (3.9) and using (3.11), (3.5), (3.13), (3.14) :

$$\begin{aligned} R_{\mu\nu} = &-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + g^{\alpha\alpha'}g^{\beta\beta'}(-\frac{1}{4}\partial_\nu g_{\alpha\beta}\partial_\mu g_{\alpha'\beta'} + \frac{1}{8}\partial_\mu g_{\beta\beta'}\partial_\nu g_{\alpha\alpha'}) \\ &+ \frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\partial_\alpha g_{\beta\mu}\partial_{\alpha'} g_{\beta'\nu} - \frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}(\partial_\alpha g_{\beta\mu}\partial_{\beta'} g_{\alpha'\nu} - \partial_{\beta'} g_{\beta\mu} - \partial_\alpha g_{\alpha'\nu}) \\ &+ \frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}((\partial_\mu g_{\alpha'\beta'}\partial_\alpha g_{\beta\nu} - \partial_\alpha g_{\alpha'\beta'} - \partial_\mu g_{\beta\nu}) + (\partial_\nu g_{\alpha'\beta'}\partial_\alpha g_{\beta\mu} - \partial_\alpha g_{\alpha'\beta'} - \partial_\nu g_{\beta\mu})) \\ &+ \frac{1}{4}g^{\alpha\alpha'}g^{\beta\beta'}((\partial_{\beta'} g_{\alpha'\alpha}\partial_\mu g_{\beta\nu} - \partial_\mu g_{\alpha'\alpha} - \partial_{\beta'} g_{\beta\nu}) + (\partial_{\beta'} g_{\alpha'\alpha}\partial_\nu g_{\beta\mu} - \partial_\nu g_{\alpha'\alpha} - \partial_{\beta'} g_{\beta\mu})) \end{aligned} \tag{3.15}$$

Finally,

$$R_{\mu\nu} = 0 \tag{3.16}$$

gives the result

□

Let $m = diag(-1,1,1,1)$ denote the minkowski metric in Cartesian coordinates and write

$$g = m + h^0 + h^1 \qquad \text{where} \qquad h^0_{\mu\nu}(t,x) = \chi(\frac{r}{t})\chi(r)\frac{M}{r}\delta_{\mu\nu} \tag{3.17}$$

where g is a solution of the reduced Einstein system.

We also define, for $0<\gamma<1$ and $0<\mu<1-\gamma$, the energy at time t,

$$E_N(t) = \sum_{|I|\le N}\|w^{1/2}\partial Z^I h^1(t,.)\|^2_{L^2}, \qquad \text{where} \qquad w(t,x) = \begin{cases} (1+|r-t|)^{1+2\gamma} & r>t \\ 1+(1+|r-t|)^{-2\mu} & r\le t \end{cases} \tag{3.18}$$

where I is a multi index and Z^I denotes a combination of $|I|$ of the following vector fields :

$$\Omega_{ij} = x^i\partial_{x^j} - x^j\partial_{x^i} \qquad B_i = x^i\partial_t + t\partial_{x^i} \qquad S = t\partial_t + x^k\partial_{x^k} \qquad \partial_\alpha$$

The vector fields lie in the tangent bundle. To any vector field, we can associate a *lift* which is a vector field lying on the tangent bundle to the tangent undle of the manifold. We will recall the general construction later.

$|.|$ denotes :

- $|x| = ((x^1)^2+(x^2)^2+(x^3)^2)^{1/2}$
- $|h(t,x)| = \sum_{\alpha\beta=0}^3(|h_{\alpha\beta}(t,x)|)$
- $|\Gamma(t,x)| = \sum_{\alpha\beta\gamma=0}^3(|\Gamma^\gamma_{\alpha\beta}(t,x)|)$

We will note $\|h(t,.)\|_{L^2} = \sum_{\alpha\beta=0}^3(\|h_{\alpha\beta}(t,.)\|_{L^2})$ and $A\lesssim B$ means that there exists a constant C such that $A\le CB$.

The main result of this work will be te prove the following theormem (3.1), and theorem (1.1) will be a corollary of this theorem.

Theorem 3.1. *For any $0<\gamma<1$ and $N\ge 11$, there exists ϵ_0 such that, for any data $(g_{\mu\nu},\partial_t g_{\mu\nu}, f)|_{t=0}$ for the reduced Einstein-Vlasov system which satisfy the smallness condition*

$$E_N(0)^{1/2} + \mathcal{V}_N + M \le \epsilon$$

for any $\epsilon\le\epsilon_0$ and the wave coordinate condition and such that $f|_{t=0}$ is complactly supported, there exists a global solution attaining the data such that

$$(E_N(t))^{\frac{1}{2}} + \sum_{|I|\le N}(1+t)\|Z^I T^{\mu\nu}(t,.)\|_{L^2} \le C_N\epsilon(1+t)^{C'_N\epsilon} \tag{3.19}$$

for all $t \geq 0$ along with the decay estimates

$$|Z^I h^1(t,x)| \leq \frac{C'_N \epsilon (1+t)^{C'_N \epsilon}}{(1+t+r)(1+q_+)^\gamma}, \qquad |I| \leq N-3, \qquad q_+ = \begin{cases} r-t, & r > t \\ 0, & r \leq t \end{cases} \tag{3.20}$$

along with the decay estimates (9.4), (9.5), (9.26), (9.38), (9.39) and (9.47).

Given $(\Sigma, \overline{g}, k, f_0)$ and initial data :

$$g_{ij}|_{t=0} = \overline{g_{ij}} \qquad g_{00}|_{t=0} = -a^2 \qquad g_{0i}|_{t=0} = 0 \qquad a(x) = (1 - \chi(r)M/r)$$

and

$$\partial_t g_{ij}|_{t=0} = -2ak_{ij} \qquad \partial_t g_{00}|_{t=0} = 2a^3 \overline{g}^{ij} k_{ij} \qquad \partial_t g_{0j}|_{t=0} = a^2 \overline{g}^{jk} \partial j \overline{g}_{ik} - \frac{a^2}{2} \overline{g}^{jk} \partial i \overline{g}_{jk} - a \partial_i a$$

one can show that $(E_N(0)^{1/2}) \lesssim \mathcal{E}_N$ and thus theorem (1.1) is a corollary of theorem 3.1 and the data satisfy the harmonic gauge.

It is important to note that the energy was defined in terms of h^1 and not in terms of $h = h^0 + h^1$. The data given then by $h_{\mu\nu}|_{t=0} = \frac{M}{r}\delta_{\mu\nu} + o(r^{-1-\gamma})$ and $\partial_t h_{\mu\nu}|_{t=0} = o(r^{-2-\gamma})$ would have appeared to be small but it does not have a sufficient decay rate in r at infinity due to the presence of the term with positive mass M. The decay of the solution at the time-like infinity would have been affected by the slow fall-off at the space-like infinity. Moreover, if the energy had been defined with h instead of h^1 we would have had an infinite energy $E_N(0)$. We can see that we need f_0 to have a compact support. Indeed throughout the proof we notice that the slowest decay for the wave equation occur in the zone $t \sim r$ whilst in the Vlasov equation they occur in the zone $r < t$, hence the assumptions on f_0.

3.3 Configuration of the problem

In this subsection, we introduce notations that will be used across all the report. The mass shell $\mathcal{P}$ will be now parameterize by $(t, x, \hat{p})$ where

$$\hat{p}^i = \frac{p^i}{p^0}$$

Let Σ_t denote the level hypersurfaces of the time coordinate t, then the solutions of the geodesic equations can be written as

$$\frac{dX^i}{ds}(s,t,x,\hat{p}) = \hat{P}^i(s,t,x,\hat{p}) \qquad \frac{d\hat{P}^i}{ds}(s,t,x,\hat{p}) = \hat{\Gamma}^i(s, X(s,t,x,\hat{p}), \hat{P}(s,t,x,\hat{p})) \tag{3.21}$$

normalised so that $(s, X(s,t,x,\hat{p})) \in \Sigma_s$ with,

$$X^i(t,t,x,\hat{p}) = x^i \qquad \hat{P}^i(t,t,x,\hat{p}) = \hat{p}^i$$

and where

$$\hat{\Gamma}^\mu(t,x,\hat{p}) = \Gamma^0_{\alpha\beta}(t,x)\hat{p}^\alpha\hat{p}^\beta\hat{p}^\mu - \Gamma^\mu_{\alpha\beta}(t,x)\hat{p}^\alpha\hat{p}^\beta$$

Let functions $\Lambda^{\alpha\beta,\mu}_\gamma$ be defined, so that,

$$\Gamma(t,x) \cdot \Lambda(\hat{p})^\mu = \hat{\Gamma}(t,x,\hat{p})^\mu$$

where

$$\Gamma(t,x) \cdot \Lambda(\hat{p})^\mu := \Gamma^\gamma_{\alpha\beta}(t,x)\Lambda^{\alpha\beta,\mu}_\gamma(\hat{p})$$

We also define $X(s,t,x,\hat{p})^0 = s$ and $\hat{P}(s,t,x,\hat{p})^0 = 1$ and we will often use the notations $X(s)$ and $\hat{P}(s)$ for $X(s,t,x,\hat{p})$ and $\hat{P}(s,t,x,\hat{p})$ when the context is clear. Finally we will use the notation $\hat{X}(s) = (s, X(s))$ The Vlasov equation can be rewritten as,

$$f(t,x,\hat{p}) = f(s, X(s,t,x,\hat{p}), \hat{P}(s,t,x,\hat{p})) = f_0(X(0,t,x,\hat{p}), \hat{P}(0,t,x,\hat{p})) \tag{3.22}$$

for all s. The notation (y, q) will be used to denote points in the mass shell over the initial hypersurface, $\mathcal{P}|_{t=0}$. Under a relatively mild smallness assumptions on $h = g - m$ we will see that there exists $c < 1$ such that solutions of the Vlasov equation satisfy

$$\operatorname{supp} f \subset \{(t,x,p); |x| \leq K + ct, |p| \leq K' + 1, |\hat{p}| \leq c\} \tag{3.23}$$

In order to prove theorem (3.1), the following theorem will be proved in the following sections :

Theorem 3.2. *For a given $t \geq 0$ and $N \geq 1$, suppose that g is a Lorentzian metric such that the Christoffel symbols of g with respect to a global coordinate system (t, x^1, x^2, x^3), for some $1/2 < a < 1$, satisfy*

$$|Z^I \Gamma(t', x)| \leq \frac{C'_N \epsilon}{(1+t')^{1+a}} \qquad \text{for } |x| \leq ct' + K, \qquad |I| \leq \left\lfloor \frac{N}{2} \right\rfloor + 2 \tag{3.24}$$

for all $t' \in [0, t]$. Then there exists $\epsilon_0 > 0$ such that, for $\epsilon < \epsilon_0$ and any solution f of the Vlasov equation satisfying (3.23), and metric satisfying (3.24) and $|g - m| \leq \epsilon$, the components of the energy momentum tensor $T^{\mu\nu}(t, x)$ satisfy

$$\|(Z^I T^{\mu\nu})(t,.)\|_{L^1} \leq D_k \mathcal{V}_k + D_k \mathbb{D}_{k'} \Big(\sum_{|J| \leq |I|-1} \frac{\|(Z^J \Gamma)(t,.)\|_{L^2}}{(1+t)^{a-1/2}} + \sum_{|J| \leq |I|+1} \int_0^t \frac{\|((Z^J \Gamma)(s,.)\|_{L^2}}{(1+s)^{1/2+a}} ds \Big),$$

for $|I| \leq N-1$, where $k = |I|$ and $k' = \lfloor \frac{k}{2} \rfloor + 1$, and

$$\|(Z^I T^{\mu\nu})(t,.)\|_{L^2} \leq \frac{D_k \mathcal{V}_k}{(1+t)^{3/2}} + D_k \mathbb{D}_{k'} \Big(\sum_{|J| \leq |I|-1} \frac{\|(Z^J \Gamma)(t,.)\|_{L^2}}{(1+t)^{1+a}} + \sum_{|J| \leq |I|} \frac{1}{(1+t)^{3/2}} \int_0^t \frac{\|((Z^J \Gamma)(s,.)\|_{L^2}}{(1+s)^{1/2}} ds \Big),$$

for $|I| \leq N$. Here the constants D_k depend only on C'_N,K,K' and c, and ϵ_0 depends only on c.

The proof of theorem 3.1 is a direct consequence of theorem 3.2 and propositions that will be shown in this report. One gives T_* the supremum of all times T_1 such that a solution of the reduced Einstein-Vlasov system attaining the given data exists for all $t \in [0, T_1]$ and satisfies the bounds :

$$E_N(t)^{1/2} \leq C_N \epsilon (1+t)^\delta, \qquad \sum_{|I| \leq N-1} \|Z^I \hat{T}(t,.)\|_{L^1} \leq C_N \epsilon \tag{3.25}$$

for all $t \in [0, T_1]$, where $\delta > 0$ and C_N is a fixed large constant. The set of such T_1 is non empty by the local existence theorem of Choquet-Bruhat. The main objective is thus to use the Einstein equations to prove that the bounds in fact hold with better constants, provided the initial data are sufficiently small.

A really important ingredient for this proof will be the energy inequality with weights :

$$\begin{aligned} \int_{\Sigma_t} |\partial\phi|^2 w dx + \int_0^t \int_{\Sigma_\tau} |\overline{\partial}\phi|^2 w' dx d\tau \leq 8 \int_{\Sigma_0} |\partial\phi|^2 w dx + \int_0^t \frac{C\epsilon}{1+\tau} \int_{\Sigma_t} |\partial\phi|^2 w dx d\tau \\ + 16 \int_0^t \Big(\int_{\Sigma_\tau} |\tilde{\Box}_g \phi|^2 w dx \Big)^{1/2} \Big(\int_{\Sigma_\tau} |\partial\phi|^2 w dx \Big)^{1/2} d\tau \end{aligned} \tag{3.26}$$

where $\overline{\partial}$ denotes the derivatives tangential to the outgoing Minkowski light cones that will be described in the subsection "The null frame". We can see that the theorem 3.2 will be mostly use in this inequality after commuting the system.

4 The support of the matter and approximations to the geodesics

In this section, we will show some properties about the support of the distribution function f (3.23) and introduce curves which approximate the geodesics.

4.1 The importance of geodesic approximation

For quasilinear system of wave equations, controlling the non-linearities requires estimates obtained after commutation of the equations with well chosen vector fields, so as to control a high number of derivatives of the solutions. One then combines these higher order estimates with weighted Sobolev inequalities to prove decay estimates for the non-linear terms. This implies that we must estimate $Z^I T(f)$ where Z^I has been defined in section 3.2 and in Minkowski's space :

$$T^{\mu\nu} f = \int_{\mathbb{R}^3} f(t,x,p) p^\mu p^\nu \frac{\sqrt{|detg|}}{p^0} dp^1 dp^2 dp^3$$

For the vector field previously introduced $Z = Z^\alpha \partial_{x_\alpha}$ we have $ZT(f) = T(Z(f))$ in Minkowski's space. But in general, if Z is defined on a Lorentzian manifold, $Z(f)$ does not really make sense since f is defined on a different manifold. Thus we need to lift Z to $\overline{Z}$ on $\mathcal{P}$. There are many ways to build the lift of a vector field on $\mathcal{P}$ based on the geodesics. Instead of the true geodesics we will use the approximations to the true geodesics. The lift's construction will be shown in other sections but it is important to note that the approximations of the geodesics will be used to avoid a regularity issue that would have appeared in theorem 3.1. We will precise the issue in the next subsections.
Before showing the proof of the support of the matter we define the approximations of the geodesics. Integrating the system (3.21) between s and t it is straightforward to show that the geodesics take the form

$$X(s,t,x,\hat{p})^k = x^k - (t-s)\hat{p}^k - \int_s^t (s'-s)\hat{\Gamma}^k(s', X(s'), \hat{P}(s'))ds'$$

Using the fact that

$$\hat{P}(s,t,x,\hat{p})^k \sim \hat{p}^k \sim \frac{x^k}{t}, \qquad X(s,t,x,\hat{p}) \sim x^k - (t-s)\hat{p}^k \sim s\frac{x^k}{t}$$

we define the approximation of the geodeics :

$$X_2(s,t,x,\hat{p})^k = x^k - (t-s)\hat{p}^k - \int_s^t (s'-s)\hat{\Gamma}^k(s', s'\frac{x}{t}, \frac{x}{t})ds'$$

4.2 The support of the matter

In theorem 3.1, we assume that f_0 is compactly supported. Under a relatively mild assumption on the metric g we will show (3.23).

Proposition 4.1. *Suppose that $|y| \leq K$ and $|q| \leq K'$ in the support of $f_0(y,q)$, for some K,$K' \geq 1$, and that for some fixed a, $\delta > 0$*

$$|g - m| \leq c' \qquad \text{and} \qquad |\Gamma| \leq \frac{c''}{(1+t)^{1-\delta}(1+|t-r|)^{a+\delta}} \tag{4.1}$$

where $c' = \frac{1}{16(1+4K'^2)}$ and $c'' = min(c'^{a+\delta}/a, (1+2K/c')^{-\delta}/\delta)/4$. Then with c=1-c' we have

$$|P(s,0,y,q)| \leq K'+1, \qquad |P(s,0,y,\hat{q})| \leq c|P^0(s,0,y,\hat{q})|, \qquad |X(s,0,y,q)| \leq cs + K, \qquad s \geq 0 \tag{4.2}$$

Proof. Let $p = P(s,0,y,q)$. Recall that $g_{\alpha\beta}p^\alpha p^\beta = -1$. It follows that

$$\begin{aligned} |1 + |p|^2 - |p^0|^2| &= |1 + m_{\alpha\beta}p^\alpha p^\beta| \\ &= |1 + (m_{\alpha\beta} - g_{\alpha\beta})p^\alpha p^\beta + \underbrace{g_{\alpha\beta}p^\alpha p^\beta}_{=-1}| \\ &= |(m_{\alpha\beta} - g_{\alpha\beta})p^\alpha p^\beta| \\ &\leq |g-m|(|p^0|^2 + |p|^2) \end{aligned}$$

and hence

$$|\frac{1}{|p^0|^2} + \frac{|p|^2}{|p^0|^2} - 1| \leq c'(1 + \frac{|p|^2}{|p^0|^2}) \Rightarrow \frac{|p|^2}{|p^0|^2}(1-c') \leq 1 + c' - |\frac{1}{|p^0|^2}|$$

$$\begin{aligned} \Rightarrow \frac{|p|^2}{|p^0|^2} &\leq \frac{1+c'}{1-c'} - \frac{1}{(1-c')|p^0|^2} \\ &\leq \frac{1+c'}{1-c'} - \frac{16(1+4K'^2)}{(15+K'^2)K'^2} \\ &\leq \frac{1+c'}{1-c'} - \frac{1}{1+4K'^2} \\ &\leq 1 + 4c' - \frac{1}{1+4K'^2} \\ &\leq (1-c')^2 = c^2 \end{aligned}$$

Thus we have

$$\frac{d}{ds}(s-|X|) = 1 - \frac{X \cdot P}{|X|P^0} \geq 1-c$$

Integrating between 0 and s we get

$$s - |X| \geq (1-c)s - K \tag{4.3}$$

Recall that $|\Gamma| \leq \frac{c''}{(1+t)^{1-\delta}(1+|t-r|)^{a+\delta}}$.
For $s \geq \frac{2K}{c'}$, along a characteristic we get :

$$|\Gamma| \leq \frac{c''}{(1+s)^{1-\delta}(1+|c's-K|)^{a+\delta}} \leq \frac{c''}{(1+s)^{1+a}} \frac{(1+s)^{a+\delta}}{(1+|c's-K|)^{a+\delta}}$$

Writing

$$\frac{1+s}{1+|c's-K|} = \frac{1}{c'}\Big(1 + \frac{K+c'-1}{1+K}\Big) \leq \frac{2}{c'}$$

we get

$$|\Gamma| \leq 2c''c'^{-a-\delta}s^{-1-a}$$

And for $s \leq \frac{2K}{c'}$, along a characteristic we have :

$$|\Gamma| \leq \frac{2c''}{(1+s)^{1-\delta}}$$

Recall the system

$$\frac{dX^i}{ds}(s,t,x,\hat{p}) = \hat{P}^i(s,t,x,\hat{p}) \qquad \frac{d\hat{P}^i}{ds}(s,t,x,\hat{p}) = \hat{\Gamma}^i(s, X(s,t,x,\hat{p}), \hat{P}(s,t,x,\hat{p}))$$

where

$$\hat{\Gamma}^\mu(t,x,\hat{p}) = \Gamma^0_{\alpha\beta}(t,x)\hat{p}^\alpha\hat{p}^\beta\hat{p}^\mu - \Gamma^\mu_{\alpha\beta}(t,x)\hat{p}^\alpha\hat{p}^\beta$$

We have

$$\frac{d}{ds}(1+|P|) = \frac{P\cdot\hat{\Gamma}}{|P|} \qquad \text{and since } \hat{p} < 1 :$$

$$\frac{d}{ds}(1+|P|) \leq |\Gamma|(1+|P|) \qquad \text{and} \qquad \frac{d}{ds}|P-q| \leq |\Gamma|(1+|P|) \tag{4.4}$$

Integrating $|\Gamma|$ from 0 to t using the previous inequalities depending on the value of s we obtain :

$$\int_0^t |\Gamma| ds \leq \int_0^{\frac{2K}{c'}} \frac{2c''}{(1+s)^{1-\delta}} ds + \int_{\frac{2K}{c'}}^\infty 2c''c'^{-a-\delta}s^{-1-a} ds \leq \frac{1}{4}$$

Then, first equation of (4.3) allows us to show that

$$\frac{1+|P|}{1+|q|} \leq e^{1/4} \leq \frac{3}{2} \qquad \text{and combined with second equation} \qquad |P-q| \leq \frac{1}{4}(1+\frac{3}{2}) \leq 1$$

It follows $|P(s,0,y,q)| \leq K'+1$ in the support of f.
The previous inequality, the inequality on $\frac{P}{P^0}$ and (4.3) show that (4.2) holds.

□

The proof for the approximation of the geodesic will also require some bounds on $|\frac{x}{t} - \hat{p}|$ and $|\hat{P}(s) - \frac{X(s)}{s}|$ for $(t,x,\hat{p})$ in $\mathrm{supp}(f)$. It is important to note that the assumptions made of Γ and the proposition 4.1 imply that

$$|Z^I\Gamma| \leq \frac{c'''_{N''}}{(1+t)^{1+a}}, \qquad \text{for} \quad |I| \leq N'', \qquad \text{where} \qquad c'''_{N''} = \frac{c''_{N''}K^{a+\delta}}{(1-c)^{a+\delta}}$$

Proposition 4.2. *If $|\Gamma(t,x)| \leq \frac{c'''}{(1+t)^{-1-a}}$ for $|x| \leq ct + K$ then, for $(t,x,\hat{p}) \in \mathrm{supp}(f)$ with $t \geq 1$, we have*

$$|\frac{x}{t} - \hat{p}| \leq (K + c''')(1+t)^{-a}$$

Proof. We can represent every point $(t, x, \hat{p}) \in \text{supp}(f)$ by a point $(y, q) \in \text{supp}(f_0)$ by writing $(t, x, \hat{p}) = (t, X(t, 0, y, q), \hat{P}(t, 0, y, q))$. Hence for $(0, y, q) \in \text{supp}(f)$:

$$|\frac{d}{ds}(X(s,0,y,q) - s\hat{P}(s,0,y,q))| = |s\hat{\Gamma}(s, X(s,0,y,q), \hat{P}(s,0,y,q))| \lesssim \frac{c'''}{(1+s)^a}$$

for all $s \geq 0$. We know that there exists C such that $|y| + |q| \leq C$ and by integrating between 0 and s :

$$|X(s,0,y,q) - s\hat{P}(s,0,y,q)| \leq K + c'''(1+s)^{1-a} \tag{4.5}$$

and the proof follows dividing by s, setting $s = t$ and using $t \geq 1$

□

4.3 Approximations to geodesics

In subsection 4.1 we have introduce the approximations to the geodesics. In this subsection we show that these curves are a good approximation in a sense we describe below. Recall that

$$X_2^i(s,t,x,\hat{p}) = x^i - (t-s)\hat{p}^i - \int_s^t (s'-s)\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})ds'$$

define the approximation of the geodesic $X^i(s, t, x, \hat{p})$ for $(t, x, \hat{p}) \in supp(f)$ and $t_0 \leq s \leq t$. We also define

$$\hat{P}_2^i(s,t,x,\hat{p}) = \hat{p}^i + \int_s^t \hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})ds'$$

for $i = 1, 2, 3$.

Finally we define

$$\overline{X}(s,t,x,\hat{p})^i = X(s,t,x,\hat{p})^i - X_2(s,t,x,\hat{p})^i$$
$$\overline{P}(s,t,x,\hat{p})^i = \hat{P}(s,t,x,\hat{p})^i - \hat{P}_2(s,t,x,\hat{p})^i$$

and note that $\overline{X}(t,t,x,\hat{p})^i = \overline{P}(t,t,x,\hat{p})^i = 0$. We also have the geodesic equations

$$\frac{d\overline{X}^i}{ds}(s,t,x,\hat{p}) = \overline{P}^i(s,t,x,\hat{p}) \tag{4.6}$$

$$\frac{d\overline{P}^i}{ds}(s,t,x,\hat{p}) = \hat{\Gamma}^i(s, X(s,t,x,\hat{p}), \hat{P}(s,t,x,\hat{p})) - \hat{\Gamma}^i(s, s\frac{x}{t}, \frac{x}{t}) \tag{4.7}$$

Proposition 4.3. *Suppose $t \geq t_0$ is such that $(t, x, \hat{p}) \in supp(f)$ and $|\Gamma(t,x)| + t|\partial\Gamma(t,x)| \leq \epsilon t^{-1-a}$. Then*

$$s^{2a-1}|\overline{X}(s,t,x,\hat{p})^i| + s^{2a}|\overline{P}(s,t,x,\hat{p})^i| \lesssim \epsilon, \tag{4.8}$$

for all $t_0 \leq s \leq t$ and i=1,2,3.

Proof. As in the previous proposition we first notice that

$$|\frac{d\hat{P}^i(s)}{ds}| \lesssim \frac{\epsilon}{s^{1+a}} \tag{4.9}$$

for $t_0 \leq s \leq t$. Then we decompose

$$|\frac{x^i}{t} - \frac{X(s)^i}{s}| \leq |\frac{x^i}{t} - \hat{p}^i| + |\hat{p}^i - \hat{P}(s)^i| + |\hat{P}(s)^i - \frac{X(s)^i}{s}|$$

We then bound the first term with Proposition 4.2, the second term with the integration of (4.9) between 0 and s and the third term with (4.5). It follows that

$$|\frac{x^i}{t} - \frac{X(s)^i}{s}| \lesssim \frac{1}{s^a} \tag{4.10}$$

Recall that

$$\frac{d\overline{P}^i}{ds}(s,t,x,\hat{p}) = \hat{\Gamma}^i(s, X(s,t,x,\hat{p}), \hat{P}(s,t,x,\hat{p})) - \hat{\Gamma}^i(s, s\frac{x}{t}, \frac{x}{t})$$

where
$$\hat{\Gamma}^\mu(t,x,\hat{p}) = \Gamma^0_{\alpha\beta}(t,x)\hat{p}^\alpha\hat{p}^\beta\hat{p}^\mu - \Gamma^\mu_{\alpha\beta}(t,x)\hat{p}^\alpha\hat{p}^\beta$$

We want to have a bound on $\frac{d\overline{P}}{ds}$ that we will integrate once to get a bound on $\overline{P}$ and integrate another time to have a bound on $\overline{X}$.

Since for $(t,x,\hat{p}) \in supp(f)$, $\frac{x}{t} \le 1$ and $\hat{p} \le 1$ we have for $t_0 \le s \le t$

$$|\hat{\Gamma}^i(s, X(s,t,x,\hat{p}), \hat{P}(s,t,x,\hat{p})) - \hat{\Gamma}^i(s, s\frac{x}{t}, \frac{x}{t}) \lesssim |\Gamma^\mu_{\alpha\beta}(s, s\frac{x}{t}) - \Gamma^\mu_{\alpha\beta}(s, X(s))||\hat{P}(s)^i - \frac{x^i}{t}|$$

From the assumption made in the proposition and (4.10) it follows

$$|\Gamma^\mu_{\alpha\beta}(s, s\frac{x}{t}) - \Gamma^\mu_{\alpha\beta}(s, X(s))| \lesssim \|\partial\Gamma^\mu_{\alpha\beta}(s,.)\|_{L^\infty}|s\frac{x}{t} - X(s)| \lesssim \frac{\epsilon}{s^{1+2a}}$$

We also have,

$$|\hat{P}(s)^i - \frac{x^i}{t}| \le |\hat{P}(s)^i - \hat{p}^i| + |\hat{p}^i - \frac{x^i}{t}| \lesssim \frac{1}{s^a}$$

For $t_0 \le s \le t$ it follows

$$|\hat{\Gamma}^i(s, X(s,t,x,\hat{p}), \hat{P}(s,t,x,\hat{p})) - \hat{\Gamma}^i(s, s\frac{x}{t}, \frac{x}{t})| \lesssim \frac{\epsilon}{s^{1+2a}}$$

Recall that $\overline{X}(t,t,x,\hat{p})^i = \overline{P}(t,t,x,\hat{p})^i = 0$

Integrating between t and s a first time and then from t and s a second time we have since $a > 1/2$:

$$|\overline{P}(s,t,x,\hat{p})^i| \lesssim \frac{\epsilon}{s^{2a}} \qquad |\overline{X}(s,t,x,\hat{p})^i| \lesssim \frac{\epsilon}{s^{2a-1}} \tag{4.11}$$

□

Corollary 4.1. Suppose $t \ge t_0$ is such that $(t,x,\hat{p}) \in supp(f)$ and $|\Gamma(t,x)| + t|\partial\Gamma(t,x)| \le \epsilon t^{-1-a}$ Then for $t_0 \le s \le t$ and i=1,2,3,
$$|X_2(s,t,x,\hat{p})^i| \lesssim s$$

Proof. The proof is an application of (4.2) and the previous proposition. □

5 Vector fields and lifts

In this section we build the lift of the vector fields :

$$\Omega_{ij} = x^i\partial_{x^j} - x^j\partial_{x^i} \qquad B_i = x^i\partial_t + t\partial_{x^i} \qquad S = t\partial_t + x^k\partial_{x^k} \qquad \partial_\alpha$$

To understand the purpose of this section we need to recall the theorem 3.2 which gives inequalities of the form

$$\|(Z^I T^{\mu\nu})(t,.)\|_{L^1} \le D_k\mathcal{V}_k + D_k\mathbb{D}_{k'}\Big(\sum_{|J|\le|I|-1}\frac{\|(Z^J\Gamma)(t,.)\|_{L^2}}{(1+t)^{a-1/2}} + \sum_{|J|\le|I|+1}\int_0^t\frac{\|((Z^J\Gamma)(s,.)\|_{L^2}}{(1+s)^{1/2+a}}ds\Big),$$

$$\|(Z^I T^{\mu\nu})(t,.)\|_{L^2} \le \frac{D_k\mathcal{V}_k}{(1+t)^{3/2}} + D_k\mathbb{D}_{k'}\Big(\sum_{|J|\le|I|-1}\frac{\|(Z^J\Gamma)(t,.)\|_{L^2}}{(1+t)^{1+a}} + \sum_{|J|\le|I|}\frac{1}{(1+t)^{3/2}}\int_0^t\frac{\|((Z^J\Gamma)(s,.)\|_{L^2}}{(1+s)^{1/2}}ds\Big),$$

Thus we need L^1 and L^2 estimates on $Z^I T^{\mu\nu}$. To have these estimates we need to build a lift of the vector field as it was discussed in subsection 4.1. In this section we provide a way of building these lifts eventhough slightly modified lifts will be used in theorem 3.2. We use the approximations of the geodesics to avoid a regularity issue that would have appeared in theorem 3.1.

5.1 Building the lift of a vector field

To any vector field on a manifold, we can associate a lift, which is a vector field lying on the tangent bundle to the tangent bundle of the manifold.

In order to give a brief overview, we assume that there is a flow Φ^t so that :

$$\frac{d\Phi^t}{dt} = Z(\Phi^t)$$

The mapping Φ^t can be naturally lifted into a mapping of the tangent bundle by the formula

$$\Phi^t_* = (\Phi^t, d\Phi^t)$$

This immediately defines a vector field $\overline{Z}$ on the tangent bundle by the formula

$$\frac{d\Phi^t_*}{dt} = \overline{Z}(\Phi^t_*)$$

The notion of the lift could also be understand using the Lie transport along curves but we do not give the details of this definition. Before beginning the construction of the lift, we first define the first approximations of the geodesics : $X_1(s,t,x,\hat{p}) = s\frac{x}{t}$ and $\hat{P}_1(s,t,x,\hat{p}) = \frac{x}{t}$ and we have :

$$\frac{dX_2}{ds}(s,t,x,\hat{p}) = \hat{P}_2(s,t,x,\hat{p}) \qquad \frac{d\hat{P}_2}{ds}(s,t,x,\hat{p}) = -\Gamma(X_1)\cdot\Lambda(\hat{P}_1) \qquad X_2(t) = x, \qquad \hat{P}_2(t) = \hat{p} \tag{5.1}$$

We define $F(\hat{P}_2) = \Gamma(X_1)\cdot\Lambda(\hat{P}_1)$.

$$|F(\hat{P}^i_2) - F(\hat{P'}^i_2)| = |\Gamma^0_{\alpha\beta}\frac{x^\alpha}{t}\frac{x^\beta}{t}(\hat{P}^i_2 - \hat{P'}^i_2)|$$

For some mild assumption on g and hence on Γ we have F Lipschitz continuous and thus we get unicity for the solutions of our system.

Thus for fixed τ, every points $(t,x,\hat{p}) \in \mathcal{P}$ with $t > \tau$ can be described uniquely by the pair of points $\{(t,x),(\tau,y)\}$ where

$$y = X_2(\tau,t,x,\hat{p})$$

is the point where the approximate geodesic X_2 emanating from (t,x) with velocity $\hat{p}$ intersects the hypersurface Σ_t.

Hence we can parameterised every $(t,x,\hat{p}) \in \mathcal{P}$ as $(t,x,\hat{p}_{X_2}(t,x,\tau,y))$. It is important to note that it is well defined since we have the unicity of geodesics that define y and $\hat{p}_{X_2}$ can be explicitly determined as we show on the next page.

For a given vector field Z on $\mathcal{M}$, let $\Phi^Z_\lambda : \mathcal{M} \to \mathcal{M}$ denote the family of diffeomorphisms, with $\Phi^Z_0 = Id$ and so that

$$\frac{d\Phi^Z_\lambda}{d\lambda} = Z|_{\Phi^Z_\lambda(t,x)}$$

The action of Φ^Z_λ on (t,x) and (τ,y) induces an action on $\mathcal{P}$ at time t, given by

$$\Phi^{Z,X_2}_{\lambda,\tau} = (\Phi^Z_\lambda(t,x), \hat{p}_{X_2}(\Phi^Z_\lambda(t,x), \Phi^Z_\lambda(\tau,y)))$$

For fixed t_0 we thus define $\overline{Z}$ by

$$\overline{Z}|_{(t,x,\hat{p})} = \frac{d\Phi^{Z,X_2}_{\lambda,\tau}(t,x,\hat{p})}{d\lambda}\Big|_{\lambda=0,\tau=t_0}$$

We have the following identities

$$X_2(\Phi^Z_\lambda(t_0,y)^0, \Phi^{Z,X_2}_{\lambda,\tau}(t,x,\hat{p}))^i = \Phi^Z_\lambda(t_0,y)^i$$

$$\hat{P}_2(\Phi^Z_\lambda(t_0,y)^0, \Phi^{Z,X_2}_{\lambda,\tau}(t,x,\hat{p}))^i = \hat{p}_{X_2}(\Phi^Z_\lambda(t,x), \Phi^Z_\lambda(t_0,y))^i$$

Taking the derivative with respect to λ at $\lambda = 0$ we get :

$$\overline{Z}|_{(t,x,\hat{p})}(X_2(t_0,t,x,\hat{p})^i) = Z^i|_{(t_0,X_2(t_0))} - Z^0|_{(t_0,X_2(t_0))}\hat{P}_2(t_0,t,x,\hat{p})^i \tag{5.2}$$

$$(5.3)\qquad \overline{Z}|_{(t,x,\hat{p})}(\hat{P}_2(t_0,t,x,\hat{p})^i) = \frac{d\hat{p}(\lambda)}{d\lambda}\Big|_{\lambda=0} - Z^0|_{(t_0,X_2(t_0))}\frac{d\hat{P}_2}{ds}(t_0,t,x,\hat{p})^i$$

where

$$\hat{p}(\lambda) = \hat{p}_{X_2}(\Phi^Z_\lambda(t,x),\Phi^Z_\lambda(t_0,y))$$

Remark 5.1. In order to see the utility of these equalities let's consider the function :

$$T'(t,x) = \int f(t,x,\hat{p})d\hat{p}$$

Bounds on ZT' will follow from bounds on $\overline{Z}f$. According to the Vlasov equation we have

$$(5.4)\quad \overline{Z}f(t,x,\hat{p}) = \overline{Z}(X(t_0,t,x,\hat{p})^i)(\partial_{x_i}f)(t_0,X(t_0),\hat{P}(t_0)) + \overline{Z}(\hat{P}(t_0,t,x,\hat{p})^i)(\partial_{\hat{p}_i}f)(t_0,X(t_0),\hat{P}(t_0))$$

Hence we need bounds on $|\overline{Z}(X(t_0,t,x,\hat{p})^i)|$ and $|\overline{Z}(\hat{P}(t_0,t,x,\hat{p})^i)|$.

We have

$$(5.5)\quad \overline{Z}f(t,x,\hat{p}) = \frac{d}{d\lambda}f(\Phi^Z_\lambda(t,x),\hat{p}_{X_2}(\Phi^Z_\lambda(t,x),\Phi^Z_\lambda(t_0,y)))\Big|_{\lambda=0} = Z^\alpha\partial_{x^\alpha}f(t,x,\hat{p}) + \frac{d}{d\lambda}f(t,x,\hat{p}(\lambda))\Big|_{\lambda=0}$$

Define

$$\Theta(t_0,t,x)^i = \int_{t_0}^t \frac{s-t_0}{t-t_0}\Gamma(s,s\frac{x}{t})\cdot\Lambda(\frac{x}{t})^i ds$$

which is the integral part in our geodesic equation for X_2.
From the equation $y = X_2(t_0,t,x,\hat{p}_{X_2})$ we have a definition of $\hat{p}_{X_2}$

$$y = x - (t-t_0)\hat{p}_{X_2} - (t-t_0)\Theta(t_0,t,x)$$

Hence

$$\hat{p}_{X_2}(t,x,t_0,y) = \frac{x-y}{t-t_0} - \Theta(t_0,t,x)$$

Then

$$\hat{p}(\lambda)^i = \frac{\Phi^Z_\lambda(t,x)^i - \Phi^Z_\lambda(t_0,y)^i}{\Phi^Z_\lambda(t,x)^0 - \Phi^Z_\lambda(t_0,y)^0} - \Theta(\Phi^Z_\lambda(t_0,y)^0,\Phi^Z_\lambda(t,x))^i$$

Recalling that $\Phi^Z_0 = Id$ and $y = x-(t-t_0)\hat{p}_{X_2} - (t-t_0)\Theta(t_0,t,x)$ we have

$$\frac{d\hat{p}(\lambda)^i}{d\lambda}\Big|_{\lambda=0} = \frac{(Z^i|_{(t,x)} - Z^i|_{(t_0,X_2(t_0))})(t-t_0) - \overbrace{(x^i-y^i)}^{(t-t_0)(\hat{p}^i+\Theta(t_0,t,x)^i)}(Z^0|_{(t,x)} - Z^0|_{(t_0,X_2(t_0))})}{(t-t_0)^2}$$
$$- Z^0|_{(t_0,X_2(t_0))}\partial_{t_0}\Theta(t_0,t,x) - Z^\mu|_{(t,x)}\partial_{x^\mu}\Theta(t_0,t,x)$$

Therefore

$$(5.6)\quad \frac{d\hat{p}(\lambda)^i}{d\lambda}\Big|_{\lambda=0} = \frac{Z^i|_{(t,x)} - Z^i|_{(t_0,X_2(t_0))}}{(t-t_0)} - (\hat{p}^i+\Theta(t_0,t,x)^i)\frac{Z^0|_{(t,x)} - Z^0|_{(t_0,X_2(t_0))}}{(t-t_0)}$$
$$- Z^0|_{(t_0,X_2(t_0))}\partial_{t_0}\Theta(t_0,t,x) - Z^\mu|_{(t,x)}\partial_{x^\mu}\Theta(t_0,t,x)$$

which is bounded provided the components of Z grow at most like t and having good bounds on the vector field Z applied to Γ. (5.5) and (5.6) will allow us to build the lifts of our vector fields. An example of this construction is shown in the next subsection.

5.2 The rotation vector field Ω_{12}

In this subsection, we apply the procedure described above to the rotation vector field Ω_{12}. Recall that $\Omega_{12} = x^1\partial_{x_2} - x^2\partial_{x_1}$ and let

$$Q(\lambda) = \begin{pmatrix} cos(\lambda) & -sin(\lambda) & 0 \\ sin(\lambda) & cos(\lambda) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The rotation of angle λ in the x^1, x^2 plane. We thus have

$$\Phi_\lambda^{\Omega_{12}}(t,x) = Q(\lambda)x = (cos(\lambda)x^1 - sin(\lambda)x^2, sin(\lambda)x^1 + cos(\lambda)x^2, x^3)$$

We can verify that $\frac{d\Phi_\lambda^{\Omega_{12}}}{d\lambda} = \Omega_{12}|_{\Phi_\lambda^{\Omega_{12}}(t,x)}$, $Q(0) = I_3$ and

$$Q'(\lambda) = \begin{pmatrix} -sin(\lambda) & -cos(\lambda) & 0 \\ cos(\lambda) & -sin(\lambda) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so

$$Q'(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It follows that for $y = (y^1, y^2, y^3)$ we have $(Q'(0)y)^i = y^1\delta_2^i - y^2\delta_1^1$ which is exactly $\frac{d\Phi_\lambda^{\Omega_{12}}}{d\lambda}\Big|_{\lambda=0} = \Omega_{12}|_{\Phi_\lambda^{\Omega_{12}}(t,x)}\Big|_{\lambda=0}$.

Recall that we have

$$\hat{p}_{X_2}(t,x,t_0,y) = \frac{x-y}{t-t_0} - \Theta(t_0,t,x)$$

Applying this equality to Qx and Qy we have

$$\hat{p}_{X_2}(t,Qx,t_0,Qy) = \frac{Qx-Qy}{t-t_0} - \Theta(t_0,t,Qx)$$

Multiplying by Q^{-1} it follows

$$Q^{-1}\hat{p} = \frac{x-y}{t-t_0} - Q^{-1}\Theta(t_0,t,Qx)$$

We then derive this identity with respect to λ at $\lambda = 0$ and we get

$$-Q'(0)\hat{p} + \hat{p}'(0) = Q'(0)\Theta(t_0,t,x) - \underbrace{\partial_x\Theta(t_0,t,x)}_{\Theta_x(t_0,t,x)}Q'(0)x$$

And finally

$$\hat{p}'(0) = Q'(0)(\Theta(t_0,t,x) + \hat{p}) - \Theta_x(t_0,t,x)Q'(0)x$$

From $\overline{Z}f(t,x,\hat{p}) = Z^\alpha\partial_{x^\alpha}f(t,x,\hat{p}) + \frac{d}{d\lambda}f(t,x,\hat{p}(\lambda))\Big|_{\lambda=0}$ (5.5) we have

$$\overline{\Omega_{12}}f(t,x,\hat{p}) = \partial_x f(t,x,\hat{p})Q'(0)x + \partial_{\hat{p}}f(t,x,\hat{p})(Q'(0)(\Theta + \hat{p}) - \Theta_x Q'(0)x)$$

Recall that

$$\Theta(t_0,t,x)^i = \int_{t_0}^t \frac{s-t_0}{t-t_0}\Gamma(s,s\frac{x}{t})\cdot\Lambda(\frac{x}{t})^i ds$$

and hence we can define

$$\Theta_x(t_0,t,x)^i Q'(0) = \Theta_{\Omega_{12}}(t_0,t,x)^i$$

where

$$\Theta_{\Omega_{12}}(t_0,t,x) = \int_{t_0}^t \frac{s-t_0}{t-t_0}(\Omega_{12}\Gamma)(s,s\frac{x}{t})\cdot\Lambda(\frac{x}{t}) + \Gamma(s,s\frac{x}{t})\cdot\Omega_{12}\Lambda(\frac{x}{t})ds$$

Finally we have

$$\overline{\Omega_{12}} = x^1\partial_{x^2} - x^2\partial_{x^1} + (\hat{p}^1 + \Theta^1)\partial_{\hat{p}^2} - (\hat{p}^2 + \Theta^2)\partial_{\hat{p}^1} - \Theta_{\Omega_{12}}^l\partial_{\hat{p}^l}$$

which is reduced in Minkowki space (where $\Theta = 0$) in

$$\overline{\Omega_{12}}^M = x^1\partial_{x^2} - x^2\partial_{x^1} + \hat{p}^1\partial_{\hat{p}^2} - \hat{p}^2\partial_{\hat{p}^1}$$

We will not describe here the procedure for the other vector fields but suffice to say that the procedure is the same for the other Ω_{ij}. For the scaling vector field S in Minkowski space we use $Q(\lambda) = (1+\lambda)I$ and for the boosts vector field in Minkowski space B we use $Q(\lambda)(t,x) = ((t+\lambda x^1)\gamma, (x^1+\lambda t)\gamma, x^2, x^3)$. We get

$$\overline{S}^M = S^\mu\partial_x\mu$$

$$\overline{B_1}^M = B_1^\mu\partial_{x_\mu} + (\delta^{k1} - \hat{p}^k\hat{p}^1)\partial_{\hat{p}^k}$$

5.3 Modification of the vector fields

Before modifying our vector fields let's give two definitions :

First we give the definition of a Killing vector field

Definition 5.1. A Killing vector field X fulfills the condition : $\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$

The vector fields used on $\mathcal{M}$ are Killing. The definition of the complete lift is the following :

Definition 5.2. Let Z be a vector field of the form $Z = Z^\alpha\partial_{x_\alpha}$, then let,

$$\overline{Z} = Z^\alpha\partial_{x_\alpha} + p^\beta\frac{\partial Z^i}{\partial x^\beta}\partial_{p^i}$$

where $(p^\beta) = (p^0, p^1, p^2, p^3)$ with $p^0 = \sqrt{1+|p|}$ be called the complete lift of Z to the submanifold corresponding to $p^0 = \sqrt{1+|p|}$.

It is important to note that this definition only apply to the submanifold corresponding to $p^0 = \sqrt{1+|p|}$

Complete lifts have the following properties :

— The complete lift operation lifts vector fields on $\mathcal{M}$ to vector fields on $T\mathcal{M}$.

— If Z is Killing for any regular distribution f defined on $\mathcal{P}$, $\overline{Z}(f)$ is well-defined.

— If Z is Killing, $\overline{Z}$ commutes with the geodesic spray vector field $\mathbf{X}$.

The vector fields $\overline{Z}^M$ described above in the Minkowski space are the complete lifts of Z. If it seems at first that working with complete lifts would thus solve the difficulties involved with commuting the Vlasov equation and the energy momentum tensor one should not expect the original Killing vector fields to remain killing. Moreover if we try to control $\overline{Z}^M f$, one can show that $\overline{Z}^M$ only satisfy

$$\overline{Z}^M f(t,x,\hat{p}) \leq Clog(t)$$

for solutions f of the Vlasov equation which cannot be used in the context of theorem 3.1. We will see an example on $\overline{\Omega_{ij}}^M$ below. Applying the rotation vector field to the approximation of the geodesic gives,

(5.7)

$$\begin{aligned}
\overline{\Omega_{ij}}^M(X_2(t_0,t,x,\hat{p})^k) &= (x^i - (t-t_0)\hat{p}^i)\delta_j^k - (x^j - (t-t_0)\hat{p}^j)\delta_i^k - \overline{\Omega_{ij}}^M\Big(\int_{t_0}^t (s'-t_0)\hat{\Gamma}^k(s', s'\frac{x}{t}, \frac{x}{t})ds'\Big) \\
&= X_2(t_0,t,x,\hat{p})^i\delta_j^k - X_2(t_0,t,x,\hat{p})^j\delta_i^k \\
&\quad + \int_{t_0}^t (s'-t_0)[\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})\delta_j^k - \hat{\Gamma}^j(s', s'\frac{x}{t}, \frac{x}{t})\delta_i^k - \Omega_{ij}(\hat{\Gamma}^k(s', s'\frac{x}{t}, \frac{x}{t}))]ds'
\end{aligned}$$

The first two terms are bounded since $\overline{X}$ is bounded and $f|_{t=t_0}$ has a compact support. Nevertheless, the last term grows in t and we need to modify the vector field by writing

$$\overline{\Omega_{ij}} = \overline{\Omega}_{ij}^M + \mathring{\Omega}_{ij}^l\partial_{\hat{p}^l}$$

where

$$\mathring{\Omega}_{ij}^l = \int_{t_0}^t \frac{s'-t_0}{t-t_0}[\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})\delta_j^k - \hat{\Gamma}^j(s', s'\frac{x}{t}, \frac{x}{t})\delta_i^k - \overline{\Omega}_{ij}^M(\hat{\Gamma}^k(s', s'\frac{x}{t}, \frac{x}{t}))]ds'$$

The new vector field is written such that the non bounded terms are removed.The use of modified vector fields is however not without drawbacks. Since the coefficients of these vector fields depend on the solution itself, they need to be estimated.

In theorem 3.2 we use a collection of slightly simpler vector fields than those described in the previous subsection but that are a bit singular for t close to t_0. Thus across this section we will make the assumption $t \geq t_0 + 1$

For the vector fields on theorem 3.2, we impose in general that $\overline{Z}$ has the following form :

$$\overline{Z} = Z^\mu \partial_{x^\mu} + \check{Z}^i \partial_{\hat{p}^i}$$

In the previous section we had $\overline{Z}|_{(t,x,\hat{p})}(X_2(t_0,t,x,\hat{p})^i) = Z^i|_{(t_0,X_2(t_0))} - Z^0|_{(t_0,X_2(t_0))}\hat{P}_2(t_0,t,x,\hat{p})^i$
We now impose

$$\overline{Z}|_{(t,x,\hat{p})}(X_2(t_0,t,x,\hat{p})^i) = Z^i|_{(t_0,X_2(t_0))} - Z^0|_{(t_0,X_2(t_0))}\hat{p}^i \tag{5.8}$$

to hold instead.

Applying $\overline{Z} = Z^\mu \partial_{x^\mu} + \check{Z}^i \partial_{\hat{p}^i}$ to $X_2(t_0,t,x,\hat{p}) = x - (t-t_0)\hat{p} - (t-t_0)\Theta(t_0,t,x)$ gives

$$\overline{Z}(X_2(t_0,t,x,\hat{p})^i) = Z^i|_{(t,x)} - Z^0|_{(t,x)}\hat{p}^i - Z((t-t_0)\Theta(t_0,t,x)^i) - (t-t_0)\check{Z}^i|_{(t,x,\hat{p})}$$

and thus we obtain

$$\check{Z}^i|_{(t,x,\hat{p})} = \frac{1}{t-t_0}(Z^i|_{(t,x)} - Z^i|_{(t_0,X_2(t_0))} - (Z^0|_{(t,x)} - Z^0|_{(t_0,X_2(t_0))})\hat{p}^i - Z((t-t_0)\Theta(t_0,t,x)^i)) \tag{5.9}$$

One can also show that

$$\overline{Z}|_{(t,x,\hat{p})}(\hat{P}_2(t_0,t,x,\hat{p})^i) = \check{Z}^i|_{(t,x,\hat{p})} - Z^\mu|_{(t,x,\hat{p})}\partial_{x^\mu}\Psi(t_0,t,x)^i$$

where

$$\Psi(t_0,t,x)^i = \int_{t_0}^t \Gamma(s, s\frac{x}{t}) \cdot \Lambda(\frac{x}{t})^i ds.$$

which is bounded, provided the components of Z grow at most like t.

5.4 The modified vector fields

For clarity purposes this subsection is introduced to gather the different modified vector fields with their expression.

First we give the expression of the rotation vector field $\overline{\Omega}_{12}$. Recall that for the rotation vector field we have $Z^0 = 0$ so we have the same expression as in the previous section. Thus

$$\overline{\Omega_{12}} = x^1\partial_{x^2} - x^2\partial_{x^1} + (\hat{p}^1 + \Theta^1)\partial_{\hat{p}^2} - (\hat{p}^2 + \Theta^2)\partial_{\hat{p}^1} - \Theta^l_{\Omega_{12}}\partial_{\hat{p}^l}$$

where

$$\Theta_{\Omega_{12}}(t_0,t,x) = \int_{t_0}^t \frac{s-t_0}{t-t_0}(\Omega_{12}\Gamma)(s, s\frac{x}{t}) \cdot \Lambda(\frac{x}{t}) + \Gamma(s, s\frac{x}{t}) \cdot \Omega_{12}\Lambda(\frac{x}{t})ds$$

and

$$\Theta(t_0,t,x)^i = \int_{t_0}^t \frac{s-t_0}{t-t_0}\Gamma(s, s\frac{x}{t}) \cdot \Lambda(\frac{x}{t})^i ds$$

For the scaling vector field we use the proposition (5.9) to get

$$\check{S}^i|_{(t,x,\hat{p})} = \frac{1}{t-t_0}(x^i - X_2(t_0)^i - (t-t_0)\hat{p}^i - S((t-t_0)\Theta(t_0,t,x))^i) = \Theta(t_0,t,x))^i - \Theta_S(t_0,t,x))^i$$

Finally we have for the boost vector field :

$$\check{B}^i_1|_{(t,x,\hat{p})} = \frac{1}{t-t_0}((t-t_0)\delta^i_1 - (x^1 - X_2(t_0)^1)\hat{p}^i - B_1((t-t_0)\Theta(t_0,t,x))^i) = \delta^{i1} - (\hat{p}^1 + \Theta(t_0,t,x))^1)\hat{p}^i - \Theta_{B_1}(t_0,t,x))^i$$

These equations can easily be extended to all vector fields and we have the following expression for our vector fields, let $\Theta(t,x)$=$\Theta(t_0,t,x)$:
For i,j=1,2,3, $i < j$ we define :

$$\overline{\Omega}_{ij}|_{(t,x,p)} = x^i\partial_{x^j} - x^j\partial_{x^i} + \hat{p}^i\partial_{\hat{p}^j} - \hat{p}^j\partial_{\hat{p}^i} + \mathring{\Omega}^k_{ij}\partial_{\hat{p}^k}$$

where

(5.10)

$$
\begin{aligned}
\mathring{\Omega}^k_{ij}|_{(t,x,p)} =&\frac{1}{t-t_0}\Big[\int_{t_0}^t (s'-t_0)\hat{\Gamma}^i(s',s'\frac{x}{t},\frac{x}{t})\delta^k_j - \hat{\Gamma}^j(s',s'\frac{x}{t},\frac{x}{t})\delta^k_i - (x^i\partial_{x^j}-x^j\partial_{x^i})\Big(\int_{t_0}^t (s'-t_0)\hat{\Gamma}^k(s',s'\frac{x}{t},\frac{x}{t})\Big)ds'\Big]\\
=&\Theta^i(t,x)\delta^k_j - \Theta^j(t,x)\delta^k_i - \Omega_{ij}(\Theta^k(t,x))
\end{aligned}
$$

Second we define

$$
\overline{B}_i|_{(t,x,p)} = x^i\partial_t + t\partial_{x^i} + (\delta^i_j - \hat{p}^i\hat{p}^j)\partial_{\hat{p}^j} + \mathring{B}^k_i\partial_{\hat{p}^k}
$$

where,

(5.11)

$$
\begin{aligned}
\mathring{B}^k_i|_{(t,x,p)} =& -\frac{1}{t-t_0}\Big[\int_{t_0}^t (s'-t_0)\hat{\Gamma}^i(s',s'\frac{x}{t},\frac{x}{t})ds'\hat{p}^k + (x^i\partial_t + t\partial_{x^i})\Big(\int_{t_0}^t (s'-t_0)\hat{\Gamma}^k(s',s'\frac{x}{t},\frac{x}{t})\Big)ds'\Big]\\
=& -\Theta^i(t,x)\hat{p}^k - B_i(\Theta^k(t,x)) - \frac{x^i}{t-t_0}\Theta^k(t,x)
\end{aligned}
$$

Finally we define

$$
\overline{S}|_{(t,x,p)} = t\partial_t + x^i\partial_{x^i} + \mathring{S}^k\partial_{\hat{p}^k}
$$

where,

$$
\begin{aligned}
\mathring{S}^k|_{(t,x,p)} =& \frac{1}{(t-t_0)}\Big[\int_{t_0}^t (s'-t_0)\hat{\Gamma}^i(s',s'\frac{x}{t},\frac{x}{t})ds' - (t\partial_t + x^i\partial_{x^i})\Big(\int_{t_0}^t (s'-t_0)\hat{\Gamma}^i(s',s'\frac{x}{t},\frac{x}{t})ds'\Big)\Big]\\
=& \Theta^k(t,x) - S(\Theta^k(t,x)) - \frac{x^t}{t-t_0}\Theta^k(t,x)
\end{aligned}
\tag{5.12}
$$

6 Vector fields applied to geodesics

In this section, high orders of vector fields applied to geodesics are estimated which will be used to estimate vector fields applied to the energy tensor in the next section. Before beginning the computation, let's recall the Grönwall inequality that will be used.

Lemma 6.1. *Grönwall inequality : For continuous functions v,a,b :*$[t_0,t]\to\mathbb{R}$*, if*

$$
v(s) \le \int_s^t a(s')v(s')ds' + b(s),
$$

for $s\in[t_0,t]$*, then*

$$
v(s) \le b(s) + \int_s^t a(s')b(s')e^{\int_s^{s'} a(s'')ds''}ds'
$$

It will be assumed in most of this section that $t\ge t_0+1$, and $|x|\le ct$, where $0<c<1$. Moreover we suppose the bounds

$$
|Z^I\Gamma^\alpha_{\beta\gamma}(t,x')| \le \frac{\epsilon}{(t')^{1+a}} \qquad for\ t_0\le t'\le t,\ \ |x'|\le ct',\ \text{ and }\ |I|\le \Big\lfloor\frac{N}{2}\Big\rfloor+2
\tag{6.1}
$$

to hold where $\frac{1}{2}<a<1$. We also recall the following definitions

$$
\begin{aligned}
\overline{X}(s,t,x,\hat{p})^i &= X(s,t,x,\hat{p})^i - X_2(s,t,x,\hat{p})^i\\
\overline{P}(s,t,x,\hat{p})^i &= \hat{P}(s,t,x,\hat{p})^i - \hat{P}_2(s,t,x,\hat{p})^i
\end{aligned}
$$

6.1 Repeated vector fields applied to the initial conditions for approximations to geodesics

When we apply $\overline{Z}^I$ to the equation (4.7) some terms of the form $X - X_1$ appear and in order to bound those terms we will write $X - X_1 = \overline{X} + X_2 - X_1$. Thus

$$\overline{Z}^I \left(s\frac{x^i}{t} - X_2(s,t,x,\hat{p})^i \right)$$

will appear on the right hand side and therefore we need to estimate it. We will see in the next proposition that it suffices to control Z^I applied to

(6.2)

$$\begin{aligned} s\frac{x^i}{t} - X_2(s,t,x,\hat{p})^i + X_2(t_0,t,x,\hat{p})^i &= s(\frac{x^i}{t} - \hat{p}^i) + t_0\hat{p}^i + \int_s^t (s'-s)\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})ds' - \int_{t_0}^t (s'-t_0)\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})ds' \\ &= s(\frac{x^i}{t} - \hat{p}^i) + t_0\hat{p}^i - \int_s^{t_0} (s'-t_0)\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})ds' - (s-t_0)\int_s^t \hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})ds' \end{aligned}$$

The following expression will also appear

$$\overline{Z}^I \left(\frac{x^i}{t} - \hat{P}^i(s,t,x,p) \right) = -\overline{Z}^I \left(\overline{P}^i(s,t,x,p) \right) + \overline{Z}^I \left(\frac{x^i}{t} - \hat{P}_2^i(s,t,x,p) \right) \tag{6.3}$$

where

$$\hat{P}_2^i(s,t,x,p) = \hat{p}^i + \int_s^t \hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})ds'$$

Proposition 6.1. *For any multi index I, there exist smooth functions Λ such that*

$$\begin{aligned} \overline{Z}^I(X_2(t_0)^i) = X_2(t_0)^j \Big(\Lambda^i_{I,j}(\hat{p}) + \sum_{k=1}^{|I|} \sum_{|J_1|+...+|J_k| \le |I|-k} \tilde{\Lambda}^{i,J_1,...,J_k}_{I,j,i_1,...,i_k}(\hat{p},t,x) Z^{J_1}(\Theta^{i_1})...Z^{J_k}(\Theta^{i_k}) \Big) \\ + \Lambda^i_I(\hat{p}) + \sum_{k=1}^{|I|} \sum_{|J_1|+...+|J_k| \le |I|-k} \tilde{\Lambda}^{i,J_1,...,J_k}_{I,i_1,...,i_k}(\hat{p},t,x) Z^{J_1}(\Theta^{i_1})...Z^{J_k}(\Theta^{i_k}) \Big) \end{aligned}$$

for $i = 1,2,3$, where $X_2(t_0)^i = X_2(t_0,t,x,p)^i$ and $\tilde{\Lambda}^{i,J_1,...,J_k}_{I,j,i_1,...,i_k}(\hat{p},t,x)$ and $\tilde{\Lambda}^{i,J_1,...,J_k}_{I,i_1,...,i_k}(\hat{p},t,x)$ satisfy

$$\tilde{\Lambda}^{i,J_1,...,J_k}_{I,j,i_1,...,i_k}(\hat{p},t,x) = \Lambda^{i,J_1,...,J_k}_{I,j,i_1,...,i_k}\Big(\hat{p}, \frac{x}{t-t_0}, \frac{t}{t-t_0}\Big), \qquad \tilde{\Lambda}^{i,J_1,...,J_k}_{I,i_1,...,i_k}(\hat{p},t,x) = \Lambda^{i,J_1,...,J_k}_{I,i_1,...,i_k}\Big(\hat{p}, \frac{x}{t-t_0}, \frac{t}{t-t_0}\Big)$$

for some smooth functions $\Lambda^{i,J_1,...,J_k}_{I,j,i_1,...,i_k}$ and $\Lambda^{i,J_1,...,J_k}_{I,i_1,...,i_k}$

Proof. We shall prove the result by induction on $|I|$. For $|I| = 1$ we prove the following result for $(t,x,p) \in \operatorname{supp}(f)$:

$$\begin{aligned} \overline{\Omega}_{ij}(X_2(t_0,t,x,\hat{p})^k) &= X_2(t_0,t,x,\hat{p})^i\delta^k_j - X_2(t_0,t,x,\hat{p})^j\delta^k_i, \\ \overline{B}_i(X_2(t_0,t,x,\hat{p})^k) &= t_0\delta^k_i - X_2(t_0,t,x,\hat{p})^i\hat{p}^k, \\ \overline{S}(X_2(t_0,t,x,\hat{p})^k) &= X_2(t_0,t,x,\hat{p})^k - t_0\hat{p}^k \end{aligned}$$

Indeed recall that

$$X_2^i(s,t,x,\hat{p}) = x^i - (t-s)\hat{p}^i - \int_s^t (s'-s)\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})ds'$$

defines the approximation of the geodesic $X^i(s,t,x,\hat{p})$ for $(t,x,\hat{p}) \in \operatorname{supp}(f)$ and $t_0 \le s \le t$ and that

$$\Theta(t_0,t,x)^i = \int_{t_0}^t \frac{s-t_0}{t-t_0}\Gamma(s, s\frac{x}{t}) \cdot \Lambda(\frac{x}{t})^i ds$$

We shall prove the result for $|I| = 1$ for $\overline{\Omega}_{ij}$, the other vector field also follow from a straight forward computation.

We have

$$\begin{aligned} \overline{\Omega}_{ij}(X_2^k) &= x^i\delta^k_j - x^j\delta^k_i - (t-s)\hat{p}^i\delta^k_j + (t-s)\hat{p}^j\delta^k_i - \mathring{\Omega}^k_{ij}(t-s) - \Omega_{ij}(\Theta^i) \\ &= X_2^i\delta^k_j - X_2^j\delta^k_i + [\Theta^i\delta^k_j - \Theta^j\delta^k_i](t-s) - \Omega_{ij}(\Theta^i) - \mathring{\Omega}^k_{ij}(t-s) \\ &= X_2^i\delta^k_j - X_2^j\delta^k_i \end{aligned}$$

We now suppose that the result holds at $|I|$ and we shall show it for $|I|+1$.
First we notice that, for any multi index J_i we have

$$\overline{Z}(Z^{J_i}(\Theta^{i_1})) = Z^{L_i}(\Theta^{i_i})$$

where $|L_i| = |J_i| + 1$. We also have the following equations

$$\overline{\Omega}_{ij}(\frac{t}{t-t_0}) = 0, \qquad \overline{\Omega}_{ij}(\frac{x^k}{t-t_0}) = \frac{x^i}{t-t_0}\delta_j^k - \frac{x^j}{t-t_0}\delta_i^k,$$

$$\overline{B}_i(\frac{t}{t-t_0}) = \frac{x^i}{t-t_0} - \frac{x^i}{t-t_0}\frac{t}{t-t_0}, \qquad \overline{B}_i(\frac{x^k}{t-t_0}) = \frac{t}{t-t_0}\delta_i^k - \frac{x^i}{t-t_0}\frac{x^k}{t-t_0},$$

$$\overline{S}(\frac{t}{t-t_0}) = \frac{t}{t-t_0} - \left(\frac{t}{t-t_0}\right)^2, \qquad \overline{S}(\frac{x^k}{t-t_0}) = \frac{x^k}{t-t_0} - \frac{x^k}{t-t_0}\frac{t}{t-t_0},$$

$$\overline{\Omega}_{ij}(\hat{p}^k) = (\hat{p}^i + \Theta^i)\delta_j^k - (\hat{p}^j + \Theta^j)\delta_i^k - \Omega_{ij}(\Theta^k) \tag{6.4}$$

$$\overline{B}_i(\hat{p}^k) = (\delta_i^k - \hat{p}^i\hat{p}^j) - \Theta^i\hat{p}^k - \frac{x^i}{t-t_0}\Theta^k - B_i(\Theta^k) \tag{6.5}$$

$$\overline{S}(\hat{p}^k) = \Theta^k - \frac{t}{t-t_0}\Theta^k - S(\Theta^k) \tag{6.6}$$

Now we have

$$\overline{Z}^I(X_2(t_0)^k) = X_2(t_0)^p \overbrace{\Big(\Lambda^k_{I,p}(\hat{p}) + \sum_{r=1}^{|I|}\sum_{|J_1|+...+|J_r|\le|I|-r} \tilde{\Lambda}^{k,J_1,...,J_r}_{I,p,k_1,...,k_r}(\hat{p},t,x) Z^{J_1}(\Theta^{k_1})...Z^{J_r}(\Theta^{k_r})\Big)}^{A}$$
$$+ \Lambda^k_I(\hat{p}) + \underbrace{\sum_{r=1}^{|I|}\sum_{|J_1|+...+|J_r|\le|I|-r} \tilde{\Lambda}^{k,J_1,...,J_r}_{I,k_1,...,k_r}(\hat{p},t,x) Z^{J_1}(\Theta^{k_1})...Z^{J_r}(\Theta^{k_r})\Big)}_{B}$$

Applying $\overline{\Omega}_{ij}$ (the same idea applies for the other vector field) to this equation we have

$$\overline{\Omega}_{ij}\overline{Z}^I(X_2(t_0)^k) = (X_2(t_0)^i\delta_j^p - X_2(t_0)^j\delta_i^p)A + X_2(t_0)^p\overline{\Omega}_{ij}(A) + \overline{\Omega}_{ij}(\Lambda^k_I(\hat{p})) + \overline{\Omega}_{ij}(B)$$

$$\overline{\Omega}_{ij}(A) = \overline{\Omega}_{ij}(\Lambda^k_{I,p}(\hat{p})) + \overline{\Omega}_{ij}(C)$$

where C denotes the sums. Let's call for $(k, |J_i|)$ fixed : $f_0 = \tilde{\Lambda}^{k,J_1,...,J_r}_{I,p,k_1,...,k_r}(\hat{p},t,x)$ and $f_u = Z^{J_u}(\Theta^{k_u})$ for $1 \le u \le r$. Then

$$\overline{\Omega}_{ij}(C) = \sum_{u=0}^{r}\sum_{r=1}^{|I|}\sum_{|J_1|+...+|J_r|\le|I|-r} \overline{\Omega}_{ij}(f_u)\prod_{s\neq u} f_s$$

For $u \ge 1$, $\overline{\Omega_{ij}}(Z^{J_u}(\Theta^{k_u})) = Z^{L_u}(\Theta^{k_u})$ where $|L_u| = |J_u| + 1$

Thus we have

$$\overline{\Omega}_{ij}\overline{Z}^I(X_2(t_0)^k) = (X_2(t_0)^i\delta_j^p - X_2(t_0)^j\delta_i^p)A + X_2(t_0)^p\Big[\overline{\Omega}_{ij}(\Lambda^k_{I,p}(\hat{p})) +$$
$$\sum_{\underline{u=1}}^{r}\sum_{r=1}^{|I|}\sum_{|J_1|+...+|J_r|\le|I|-r} \tilde{\Lambda} Z^{J_1}(\Theta^{k_1})...Z^{\underline{L_u}}(\Theta^{k_u})...Z^{J_r}(\Theta^{k_r}) + \sum_{r=1}^{|I|}\sum_{|J_1|+...+|J_r|\le|I|-r} \overline{\Omega}_{ij}(\tilde{\Lambda}) Z^{J_1}(\Theta^{k_1})...Z^{J_r}(\Theta^{k_r})\Big]$$
$$+ \overline{\Omega}_{ij}(\Lambda^k_I(\hat{p})) + \overline{\Omega}_{ij}(B)$$

where $\tilde{\Lambda} = \tilde{\Lambda}^{k,J_1,\dots,J_r}_{I,p,k_1,\dots,k_r}(\hat{p},t,x)$ and we have splitted the sum appearing in $\overline{\Omega}_{ij}(C)$ in two parts. Now $\overline{\Omega}_{ij}(B)$ gives the sames sums than $\overline{\Omega}_{ij}(A)$ where we would have $\tilde{\Lambda} = \tilde{\Lambda}^{k,J_1,\dots,J_r}_{I,k_1,\dots,k_r}(\hat{p},t,x)$. Hence we have

$$\overline{Z}^{I+1}(X_2(t_0)^k) = X_2(t_0)^p\Big(\Lambda^k_{I+1,p}(\hat{p}) + \sum_{r=1}^{|I|+1}\sum_{|J_1|+\dots+|J_r|\le|I|+1-r}\tilde{\Lambda}^{k,J_1,\dots,J_r}_{I+1,p,k_1,\dots,k_r}(\hat{p},t,x)Z^{J_1}(\Theta^{k_1})\dots Z^{J_r}(\Theta^{k_r})\Big)$$
$$+\Lambda^k_{I+1}(\hat{p}) + \sum_{r=1}^{|I|+1}\sum_{|J_1|+\dots+|J_r|\le|I|+1-r}\tilde{\Lambda}^{k,J_1,\dots,J_r}_{I+1,k_1,\dots,k_r}(\hat{p},t,x)Z^{J_1}(\Theta^{k_1})\dots Z^{J_r}(\Theta^{k_r})\Big)$$

where

$$\Lambda^k_{I+1,p}(\hat{p}) = \overline{\Omega}_{ij}(\Lambda^k_{I,p}(\hat{p})) + \begin{cases}\Lambda^k_{I,j}(\hat{p}) & \text{if } p=i\\ -\Lambda^k_{I,i}(\hat{p}) & \text{if } p=j\end{cases}$$

$$\tilde{\Lambda}^{k,J_1,\dots,J_r}_{I+1,p,k_1,\dots,k_r}(\hat{p},t,x) = \sum_{q=1}^{k}\tilde{\Lambda}^{k,J_1,\dots,J_{q-1},\tilde{L}_q,J_q,\dots,J_r}_{I,p,k_1,\dots,k_r}(\hat{p},t,x) + \begin{cases}\overline{\Omega}_{ij}\tilde{\Lambda}^{k,J_1,\dots,J_r}_{I,p,k_1,\dots,k_r}(\hat{p},t,x) & \text{if } |J_1|+\dots+|J_r|\le|I|-r\\ \tilde{\Lambda}^{k,J_1,\dots,J_r}_{I,i,k_1,\dots,k_r}(\hat{p},t,x) & \text{if } |J_1|+\dots+|J_r|=0 \text{ and } p=j\\ -\tilde{\Lambda}^{k,J_1,\dots,J_r}_{I,j,k_1,\dots,k_r}(\hat{p},t,x) & \text{if } |J_1|+\dots+|J_r|=0 \text{ and } p=i\end{cases}$$

where $|\tilde{L}_q| = |I_q| - 1$
This ends the induction.

□

Proposition 6.2. *For any multi index I, there exist smooth functions* Λ *such that*

$$\overline{Z}^I\Big(\frac{x^i}{t}-\hat{p}^i\Big) = \Big(\frac{x^j}{t}-\hat{p}^j\Big)\Lambda^i_{I,j}(\frac{x}{t},\hat{p}) + \sum_{k=1}^{|I|}\sum_{|J_1|+\dots+|J_k|\le|I|-k+1}\tilde{\Lambda}^{i,J_1,\dots,J_k}_{I,i_1,\dots,i_k}(\hat{p},t,x)Z^{J_1}(\Theta^{i_1})\dots Z^{J_k}(\Theta^{i_k})$$

for $i=1,2,3$, *where*

$$\tilde{\Lambda}^{i,J_1,\dots,J_k}_{I,i_1,\dots,i_k}(\hat{p},t,x) = \Lambda^{i,J_1,\dots,J_k}_{I,i_1,\dots,i_k}\Big(\hat{p},\frac{x}{t},\frac{x}{t-t_0},\frac{t}{t-t_0}\Big)$$

for some smooth functions $\Lambda^{i,J_1,\dots,J_k}_{I,i_1,\dots,i_k}$.

Proof. For $|I|=1$ we have

$$\overline{\Omega}_{ij}\Big(\frac{x^k}{t}-\hat{p}^k\Big) = \Big(\frac{x^i}{t}-\hat{p}^i\Big)\delta^k_j - \Big(\frac{x^j}{t}-\hat{p}^j\Big)\delta^k_i - \mathring{\Omega}^k_{ij}$$

$$\overline{B}_i\Big(\frac{x^k}{t}-\hat{p}^k\Big) = -\Big(\frac{x^i}{t}-\hat{p}^i\Big)\hat{p}^k - \frac{x^i}{t}\Big(\frac{x^j}{t}-\hat{p}^j\Big) - \mathring{B}^k_i$$

$$\overline{S}\Big(\frac{x^k}{t}-\hat{p}^k\Big) = \mathring{S}^k$$

We now suppose that the results holds for $|I|$. Applying the rotation vector field $\overline{\Omega}_{ij}$ (the same idea for the other vector fields) we have

$$\overline{\Omega}_{ij}(\overline{Z}^I\Big(\frac{x^k}{t}-\hat{p}^k\Big)) = \overline{\Omega}_{ij}\Big[\Big(\frac{x^p}{t}-\hat{p}^p\Big)\Lambda^k_{I,p}(\frac{x}{t},\hat{p}) + \overbrace{\sum_{r=1}^{|I|}\sum_{|J_1|+\dots+|J_r|\le|I|-r+1}\tilde{\Lambda}^{k,J_1,\dots,J_k}_{I,k_1,\dots,k_r}(\hat{p},t,x)Z^{J_1}(\Theta^{k_1})\dots Z^{J_r}(\Theta^{k_r})}^{A}\Big]$$
$$= \Big[\Big(\frac{x^i}{t}-\hat{p}^i\Big)\delta^p_j - \Big(\frac{x^j}{t}-\hat{p}^j\Big)\delta^p_i - \mathring{\Omega}^p_{ij}\Big]\Lambda + \Big(\frac{x^p}{t}-\hat{p}^p\Big)\overline{\Omega}_{ij}(\Lambda) + \overline{\Omega}_{ij}(A)$$

We have already shown in the precedent proof the value of $\overline{\Omega}_{ij}(A)$. The results follows from the straightforward induction. □

Proposition 6.3. *For any multi index I, there exist smooth functions* Λ *such that*

$$\overline{Z}^I(\hat{p}^i) = \Lambda^i_I(\hat{p}) + \sum_{k=1}^{|I|}\sum_{|J_1|+\dots+|J_k|\le|I|-k+1}\tilde{\Lambda}^{i,J_1,\dots,J_k}_{I,i_1,\dots,i_k}(\hat{p},t,x)Z^{J_1}(\Theta^{i_1})\dots Z^{J_k}(\Theta^{i_k})$$

for $i = 1, 2, 3$, where

$$\tilde{\Lambda}^{i,J_1,\dots,J_k}_{I,i_1,\dots,i_k}(\hat{p}, t, x) = \Lambda^{i,J_1,\dots,J_k}_{I,i_1,\dots,i_k}\left(\hat{p}, \frac{x}{t-t_0}, \frac{t}{t-t_0}\right)$$

for some smooth functions $\Lambda^{i,J_1,\dots,J_k}_{I,i_1,\dots,i_k}$.

Proof. The result for $|I| = 1$ comes from (6.4), (6.5) and (6.6). The induction is straightforward as in the proposition 4.1 and 4.2. □

Now that we have reformulate the different components of the vector field applied to the system, it appears that we have to estimate $Z^I \Theta$.

6.2 Preliminary estimates for repeated vector fields applied to approximations to geodesics

Proposition 6.4. *Suppose $t \geq t_0 + 1$, $|x| \leq ct$ and the bounds (6.1) hold. Then for $|I| \leq N$,*

$$|Z^I \Theta(t,x)| \lesssim \sum_{|J| \leq |I|-1} t|(Z^J \Gamma)(t,x)| + \sum_{|J| \leq |I|} \frac{1}{t-t_0} \int_{t_0}^{t} (s'-t_0)|(Z^J\Gamma)(s', s'\frac{x}{t})|ds'$$

Moreover, if $|I| \leq \left\lfloor \frac{N}{2} \right\rfloor + 2$, then

$$|Z^I \Theta(t,x)| \leq \frac{C}{t^a}$$

Proof. Recall that

$$\Theta^i(t,x) = \frac{1}{t-t_0} \int_{t_0}^{t} (s'-t_0)\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})ds'$$

Thus

$$\begin{aligned} Z(\Theta^i(t,x)) &= -\frac{Z(t)}{(t-t_0)^2} \int_{t_0}^{t} (s'-t_0)\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})ds' + \frac{1}{t-t_0} Z(t) \int_{t_0}^{t} (s'-t_0)\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})ds' \\ &= -\frac{Z(t)}{t-t_0}\Theta^i(t,x) + Z(t)\hat{\Gamma}^i(t, x, \frac{x}{t}) + \frac{1}{t-t_0}\int_{t_0}^{t} (s'-t_0) Z(\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t}))ds' \end{aligned}$$

Moreover we have

$$\begin{aligned} \Omega_{ij}\left(\Gamma^{\mu}_{\alpha\beta}(s, s\frac{x}{t})\right) &= (\partial_{x^k}\Gamma^{\mu}_{\alpha\beta})\Omega_{ij}(s\frac{x^k}{t})(s, s\frac{x}{t}) \\ &= s\frac{x^i}{t}(\partial_{x^j}\Gamma^{\mu}_{\alpha\beta})(s, s\frac{x}{t}) - s\frac{x^j}{t}(\partial_{x^i}\Gamma^{\mu}_{\alpha\beta})(s, s\frac{x}{t}) \\ &= \left(\Omega_{ij}\Gamma^{\mu}_{\alpha\beta}\right)(s, s\frac{x}{t}) \end{aligned}$$

and straightforward computation also show

$$B_i\left(\Gamma^{\mu}_{\alpha\beta}(s, s\frac{x}{t})\right) = \left(B_i\Gamma^{\mu}_{\alpha\beta}\right)(s, s\frac{x}{t}) - \frac{x^i}{t}\left(S\Gamma^{\mu}_{\alpha\beta}\right)(s, s\frac{x}{t})$$

$$S\left(\Gamma^{\mu}_{\alpha\beta}(s, s\frac{x}{t})\right) = 0$$

Now for $|I| = 1$:

$$\Omega_{ij}(\Theta^k(t,x)) = \frac{1}{t-t_0}\int_{t_0}^{t} (s'-t_0)\Omega_{ij}\left(\hat{\Gamma}^k(s', s'\frac{x}{t}, \frac{x}{t})\right)ds'$$

and

$$\begin{aligned} \Omega_{ij}\left(\hat{\Gamma}^k(s', s'\frac{x}{t}, \frac{x}{t})\right) &= \Omega_{ij}\left(\Gamma^0_{\alpha\beta}(s', s'\frac{x}{t})\frac{x^{\alpha}}{t}\frac{x^{\beta}}{t}\frac{x^k}{t} - \Gamma^k_{\alpha\beta}(s', s'\frac{x}{t})\frac{x^{\alpha}}{t}\frac{x^{\beta}}{t}\right) \\ &\leq \frac{|x|^3}{t^3}\Omega_{ij}\left(\Gamma^0_{\alpha\beta}(s', s'\frac{x}{t})\right) + \frac{|x|^2}{t^2}\Omega_{ij}\left(\Gamma^k_{\alpha\beta}(s', s'\frac{x}{t})\right) \\ &\leq c^3\Omega_{ij}\left(\Gamma^0_{\alpha\beta}(s', s'\frac{x}{t})\right) + c^2\Omega_{ij}\left(\Gamma^k_{\alpha\beta}(s', s'\frac{x}{t})\right) \\ &\lesssim \sum_{|J|\leq 1} |(Z^J\Gamma)(s', s'\frac{x}{t})| \end{aligned}$$

If $|I| \leq \left\lfloor \frac{N}{2} \right\rfloor + 2$ we have

$$\Omega_{ij}\Big(\hat{\Gamma}^k(s', s'\frac{x}{t}, \frac{x}{t})\Big) \lesssim \sum_{|J|\leq 1} \frac{\epsilon}{(t)^{1+a}} \leq \frac{C}{t^a}$$

The result for the other vector fields can be shown in a similar way. The proof for $|I| \geq 2$ is a straightforward induction which concludes the proof.

□

Now that we have some bounds on $|Z^I\Theta(t,x)|$ we shall show the following proposition

Proposition 6.5. *Suppose $t \geq t_0 + 1$, $|x| \leq ct$ and the bounds (6.1) hold. Then for $|I| \leq N$,*

$$\left|\overline{Z}^I(X_2(t_0)^i)\right| \lesssim 1 + \sum_{|J|\leq|I|-1} t|(Z^J\Gamma)(t,x)| + \sum_{|J|\leq|I|} \frac{1}{t-t_0}\int_{t_0}^t (s'-t_0)|(Z^J\Gamma)(s', s'\frac{x}{t})|ds' \tag{6.7}$$

$$\left|\overline{Z}^I\Big(\frac{x^i}{t} - \hat{p}^i\Big)\right| \lesssim \frac{1}{t^a} + \sum_{|J|\leq|I|-1} t|(Z^J\Gamma)(t,x)| + \sum_{|J|\leq|I|} \frac{1}{t-t_0}\int_{t_0}^t (s'-t_0)|(Z^J\Gamma)(s', s'\frac{x}{t})|ds' \tag{6.8}$$

$$\left|\overline{Z}^I(\hat{p}^i)\right| \lesssim 1 + \sum_{|J|\leq|I|-1} t|(Z^J\Gamma)(t,x)| + \sum_{|J|\leq|I|} \frac{1}{t-t_0}\int_{t_0}^t (s'-t_0)|(Z^J\Gamma)(s', s'\frac{x}{t})|ds' \tag{6.9}$$

$$\left|\overline{Z}^I\Big(\int_{t_0}^s (s'-t_0)\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})ds'\Big)\right| \lesssim \sum_{|J|\leq|I|} \int_{t_0}^s (s'-t_0)|(Z^J\Gamma)(s', s'\frac{x}{t})|ds' \tag{6.10}$$

$$\left|\overline{Z}^I\Big(\int_s^t \hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t})ds'\Big)\right| \lesssim \sum_{|J|\leq|I|-1} t|(Z^J\Gamma)(t,x)| + \sum_{|J|\leq|I|} \int_s^t |(Z^J\Gamma)(s', s'\frac{x}{t})|ds' \tag{6.11}$$

for $i = 1,2,3$. In particular, for $|I| \leq \left\lfloor \frac{N}{2} \right\rfloor + 2$,

$$\left|\overline{Z}^I(X_2(t_0)^i)\right| + \left|\overline{Z}^I(\hat{p}^i)\right| \leq C, \qquad \left|\overline{Z}^I\Big(\frac{x^i}{t} - \hat{p}^i\Big)\right| \leq \frac{C}{t^a}$$

for i=1,2,3

Proof. Recall proposition 4.2 and corollary 4.1 :

— If $|\Gamma(t,x)| \leq \frac{c'''}{(1+t)^{-1-a}}$ for $|x| \leq ct + K$ then, for $(t,x,\hat{p}) \in supp(f)$ with $t \geq 1$, we have

$$|\frac{x}{t} - \hat{p}| \leq (K + c''')(1+t)^{-a}$$

— Suppose $t \geq t_0$ is such that $(t,x,\hat{p}) \in supp(f)$ and $|\Gamma(t,x)| + t|\partial\Gamma(t,x)| \leq \epsilon t^{-1-a}$ Then for $t_0 \leq s \leq t$ and i=1,2,3,

$$|X_2(s,t,x,\hat{p})^i| \lesssim s$$

Their use along with the use of proposition 6.1, 6.2, 6.3, 6.4 show that the bounds (6.7), (6.8), (6.9) hold. (6.10) and (6.11) can be show as in the proof of proposition (6.4) above.

□

Corollary 6.1. Suppose $t \geq t_0 + 1$, $|x| \leq ct$ and the bounds (6.1) hold. Then for $|I| \leq N$,

$$\left|\overline{Z}^I\Big(\frac{X_2^i(s,t,x,\hat{p})}{s}\Big)\right| \lesssim 1 + \sum_{|J|\leq|I|-1} t|(Z^J\Gamma)(t,x)| + \sum_{|J|\leq|I|} \int_{t_0}^t \frac{s'}{s'+s}|(Z^J\Gamma)(s', s'\frac{x}{t})|ds' \tag{6.12}$$

$$\left|\overline{Z}^I\Big(\frac{X_2^i(s,t,x,\hat{p})}{s} - \frac{x^i}{t}\Big)\right| \lesssim \frac{1}{s^a} + \sum_{|J|\leq|I|-1} t|(Z^J\Gamma)(t,x)| + \sum_{|J|\leq|I|} \int_{t_0}^t \frac{s'}{s'+s}|(Z^J\Gamma)(s', s'\frac{x}{t})|ds' \tag{6.13}$$

for all $t_0 \leq s \leq t$, $i = 1, 2, 3$. In particular for $|I| \leq \left\lfloor \frac{N}{2} \right\rfloor + 2$,

$$\left|\overline{Z}^I \Big(\frac{X_2^i(s,t,x,\hat{p})}{s} - \frac{x^i}{t} \Big)\right| \lesssim \frac{1}{s^a} \tag{6.14}$$

Moreover

$$\left|\overline{Z}^I \Big(\frac{x^i}{t} - \hat{P}_2^i(s,t,x,\hat{p}) \Big)\right| \lesssim \frac{1}{t^a} + \sum_{|J| \leq |I|-1} t|(Z^J\Gamma)(t,x)| + \sum_{|J|\leq|I|} \int_{t_0}^t \frac{s'}{s'+s} |(Z^J\Gamma)(s', s'\frac{x}{t})| ds' \tag{6.15}$$

In particular, for $|I| \leq \left\lfloor \frac{N}{2} \right\rfloor + 2$,

$$\left|\overline{Z}^I \Big(\frac{x^i}{t} - \hat{P}_2^i(s,t,x,\hat{p}) \Big)\right| \lesssim \frac{1}{s^a} \tag{6.16}$$

Proof. We write

$$\begin{aligned} X_2^i(s,t,x,\hat{p}) &= X_2^i(t_0,t,x,\hat{p}) + (X_2^i(s,t,x,\hat{p}) - X_2^i(t_0,t,x,\hat{p})) \\ &= X_2^i(t_0,t,x,\hat{p}) + (s-t_0)\hat{p}^i + (s-t_0)\int_s^t \hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t}) ds' + \int_{t_0}^s (s'-t_0)\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t}) ds' \end{aligned}$$

The proposition 6.5 allows us to show that,

$$\begin{aligned} \left|\overline{Z}^I \Big(X_2^i(s,t,x,\hat{p}) \Big)\right| \leq Cs\Big[1 + \sum_{|J|\leq|I|-1} t|(Z^J\Gamma)(t,x)| + \sum_{|J|\leq|I|} \frac{1}{t-t_0}\int_{t_0}^t (s'-t_0)\Big|(Z^J\Gamma)(s', s'\frac{x}{t})\Big| ds'\Big] \\ + Cs \sum_{|J|\leq|I|} \int_s^t |(Z^J\Gamma)(s', s'\frac{x}{t})| ds' + C\sum_{|J|\leq|I|} \int_{t_0}^s (s'-t_0)|(Z^J\Gamma)(s', s'\frac{x}{t})| ds' \end{aligned}$$

Dividing by s :

$$\left|\overline{Z}^I \Big(\frac{X_2^i(s,t,x,\hat{p})}{s} \Big)\right| \lesssim \Big[1 + \sum_{|J|\leq|I|-1} t|(Z^J\Gamma)(t,x)| + \sum_{|J|\leq|I|} \frac{1}{t-t_0}\int_{t_0}^t (s'-t_0)\Big|(Z^J\Gamma)(s', s'\frac{x}{t})\Big| ds'\Big]$$

Using the fact that

$$\begin{aligned} \int_{t_0}^t \frac{s'-t_0}{t-t_0} |(Z^J\Gamma)(s', s'\frac{x}{t})| &= \int_s^t \frac{s'-t_0}{t-t_0} |(Z^J\Gamma)(s', s'\frac{x}{t})| ds' + \int_{t_0}^s \frac{s'-t_0}{t-t_0} |(Z^J\Gamma)(s', s'\frac{x}{t})| ds' \\ &\lesssim \int_s^t |(Z^J\Gamma)(s', s'\frac{x}{t})| ds' + \int_{t_0}^s \frac{s'}{s} |(Z^J\Gamma)(s', s'\frac{x}{t})| \\ &\lesssim \int_{t_0}^t |(Z^J\Gamma)(s', s'\frac{x}{t})| \frac{s'}{s'+s} ds' \end{aligned}$$

(6.12) follows. For (6.13) we write :

$$\begin{aligned} \frac{X_2^i(s,t,x,\hat{p})}{s} - \frac{x^i}{t} = (\hat{p}^i - \frac{x^i}{t}) - \frac{t_0}{s}\hat{p}^i + \frac{X_2^i(t_0,t,x,\hat{p})}{s} + \frac{(s-t_0)}{s}\hat{p}^i \\ + (s-t_0)\int_s^t \hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t}) ds' + \frac{1}{s}\int_{t_0}^s (s'-t_0)\hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t}) ds' \end{aligned}$$

and proposition 6.5 allow us to conclude. For (6.15) we also use proposition 6.5 writing

$$\frac{x^i}{t} - \hat{P}_2^i(s,t,x,\hat{p}) = \frac{x^i}{t} - \hat{p}^i - \int_s^t \hat{\Gamma}^i(s', s'\frac{x}{t}, \frac{x}{t}) ds'$$

For (6.14) and (6.16) we use the bound (6.1) writing

$$\int_{t_0}^t \frac{s'}{s'+s} |(Z^J\Gamma)(s', s'\frac{x}{t})| ds' \lesssim \int_s^t |(Z^J\Gamma)(s', s'\frac{x}{t})| ds' + \int_s^{t_0} \frac{s'}{s} |(Z^J\Gamma)(s', s'\frac{x}{t})| ds'$$

$\square$

6.3 Repeated vector fields applied to difference

The aim of the section 6 is to have bounds on $|Z^I\overline{X}|$ and $|Z^I\overline{P}|$ where

$$\overline{X}(s,t,x,\hat{p})^i = X(s,t,x,\hat{p})^i - X_2(s,t,x,\hat{p})^i$$
$$\overline{P}(s,t,x,\hat{p})^i = \hat{P}(s,t,x,\hat{p})^i - \hat{P}_2(s,t,x,\hat{p})^i$$

In order to have them we must apply the vector fields to our system and thus to the difference

$$\hat{\Gamma}(X,\hat{P}) - \hat{\Gamma}(X_1,\hat{P}_1) = (\Gamma(X)-\Gamma(X_1))\cdot\Lambda(\hat{P}) + \Gamma(X_1)\cdot(\Lambda(\hat{P})-\Lambda(\hat{P}_1)) \tag{6.17}$$

This expression leads us to find some bounds of the vector fields applied to $\Lambda(\hat{P})-\Lambda(\hat{P}_1)$ and $\Gamma(X)-\Gamma(X_1)$. Since we can split $X-X_1 = \overline{X}+X_2-X_1$ and $\hat{P}-\hat{P}_1 = \overline{P}+\hat{P}_2-\hat{P}_1$ and given the bounds of the last section we need to express the two differences above as a linear in the $X-X_1$ or $\hat{P}-\hat{P}_1$. Therefore we show the following lemma :

Lemma 6.2. *Given $Y_1,...,Y_k \in \mathbb{R}^3$ and $F:\mathbb{R}^3\to\mathbb{R}$, let $Y_1\cdots Y_k\cdot(\partial^k F)(Y)$ denote the sum of $Y_1^{j^1}\cdots Y_k^{j^k}\cdot(\partial_{j_1}\cdots F)(Y)$ over all components $1\le j_1\le n$ for $i=1,...,k$. We have*

(6.18)
$$\begin{aligned}
\overline{Z}^J(F(X)-F(X_1)) = &\sum_{k<|J|/2,\ J_1+...+J_k=J,\ 1\le|J_1|\le...\le|J_k|} c_{J_1...J_k}\overline{Z}^{J_1}X_1\cdots\overline{Z}^{J_k}X_1\cdot((\partial^kF)(X)-(\partial^kF)(X_1))\\
&+\sum_{k<|J|/2,\ J_1+...+J_k=J,\ 1\le|J_1|\le...\le|J_k|} c_{J_1...J_k}\sum_{1\le l\le k}\overline{Z}^{J_1}X\cdots\overline{Z}^{J_{l-1}}X\cdot\overline{Z}^{J_l}(X-X_1)\overline{Z}^{J_{l+1}}X_1\cdots\overline{Z}^{J_k}X_1\cdot(\partial^kF)(X)\\
&+\sum_{k\ge|J|/2,\ J_1+...+J_k=J,\ 1\le|J_1|\le...\le|J_k|} c'_{J_1...J_k}\overline{Z}^{J_1}X\cdots\overline{Z}^{J_k}X\cdot((\partial^kF)(X)-(\partial^kF)(X_1))\\
&+\sum_{k\ge|J|/2,\ J_1+...+J_k=J,\ 1\le|J_1|\le...\le|J_k|} c'_{J_1...J_k}\sum_{1\le l\le k}\overline{Z}^{J_1}X_1\cdots\overline{Z}^{J_{l-1}}X_1\cdot\overline{Z}^{J_l}(X-X_1)\overline{Z}^{J_{l+1}}X\cdots\overline{Z}^{J_k}X\cdot(\partial^kF)(X_1)
\end{aligned}$$

The expression above will be applied for $F=\Lambda$ and $F=\Gamma$. We know that for $|J_i|\le|J|/2$ we will be able to estimate the factors $\overline{Z}^{J_i}X$ by their L^∞ norms. Note that the sums have been splitted according to two events, whether $k\ge|J|/2$ or not. Depending on k we will use L^∞ norms either on $\partial^k(F)$ and $\partial^{k+1}(F)$ or on the other factors.

Proof. There exists $c_{J_1...J_k}$ such that

$$\overline{Z}^J(F(X)-F(X_1)) = \sum_{k,\ J_1+...+J_k=J,\ 1\le|J_1|\le...\le|J_k|} c_{J_1...J_k}\overline{Z}^{J_1}X\cdots\overline{Z}^{J_k}X\cdot(\partial^kF)(X)-\overline{Z}^{J_1}X_1\cdots\overline{Z}^{J_k}X_1\cdot(\partial^kF)(X_1)$$

Now we make a different composition depending on the size of k. If $k\le|J|/2$ one writes

$$\begin{aligned}
&\overline{Z}^{J_1}X\cdots\overline{Z}^{J_k}X\cdot(\partial^kF)(X)-\overline{Z}^{J_1}X_1\cdots\overline{Z}^{J_k}X_1\cdot(\partial^kF)(X_1)\\
&\quad=\overline{Z}^{J_1}X\cdots\overline{Z}^{J_k}X\cdot((\partial^kF)(X)-\partial^kF(X_1))+(\overline{Z}^{J_1}X\cdots\overline{Z}^{J_k}X-\overline{Z}^{J_1}X_1\cdots\overline{Z}^{J_k}X_1)\cdot(\partial^kF)(X_1)
\end{aligned}$$

And now writing

$$\begin{aligned}
\overline{Z}^{J_1}X\cdots\overline{Z}^{J_k}X &= \overline{Z}^{J_1}X\cdots\overline{Z}^{J_k}(X-X_1)+\overline{Z}^{J_1}X\cdots\overline{Z}^{J_k}X_1\\
&=\overline{Z}^{J_1}X\cdots\overline{Z}^{J_k}(X-X_1)+\overline{Z}^{J_1}X\cdots\overline{Z}^{J_{k-1}}(X-X_1)\overline{Z}^{J_k}X_1+\overline{Z}^{J_1}X\cdots\overline{Z}^{J_{k-1}}(X_1)\overline{Z}^{J_k}X_1\\
&=...\\
&=\overline{Z}^{J_1}X_1\cdots\overline{Z}^{J_k}X_1+\sum_{1\le l\le k}\overline{Z}^{J_1}X\cdots\overline{Z}^{J_{l-1}}X\cdot\overline{Z}^{J_l}(X-X_1)\overline{Z}^{J_{l+1}}X_1\cdots\overline{Z}^{J_k}X_1
\end{aligned}$$

This produces the first two sums with $k<|J|/2$. For $k\ge|J|/2$ w do the same thing but with X_1 instead of X.

□

We can now apply this lemma to Λ. We obtain

Lemma 6.3. *Suppose $|Z^K \hat{P}| \leq C$ for $|K| \leq |L|/2$. Then*

$$|\overline{Z}^L(\Lambda(\hat{P}) - \Lambda(\hat{P}_1))| \lesssim \sum_{|M| \leq |L|} |\overline{Z}^M(\hat{P} - \hat{P}_1)| \tag{6.19}$$

$$|\overline{Z}^L(\Lambda(\hat{P}_1))| \lesssim 1 \tag{6.20}$$

Proof. Note that $|\overline{Z}^I(\hat{P}_1)| \lesssim 1$ for any I and that we have the following estimate

$$|(\partial^k \Lambda)(\hat{P}) - (\partial^k \Lambda)(\hat{P}_1)| \lesssim |\hat{P} - \hat{P}_1|$$

The bound (6.19) comes from the previous lemma and the other bound is straightforward. □

Lemma 6.4. *Suppose $|Z^K X| \leq Cs$ for $|K| \leq |K|/2$. Then*

$$|\overline{Z}^J(\Gamma(X) - \Gamma(X_1))| \lesssim \sum_{|K| \leq |J|/2} |(Z^K \Gamma)(X)| \sum_{|M| \leq |J|} \frac{|\overline{Z}^M(X - X_1)|}{s} + \sum_{|M| \leq |J|} |(Z^M \Gamma)(X) - (Z^M \Gamma)(X_1)|$$
$$+ \sum_{|K| \leq |J|/2} \frac{|\overline{Z}^K(X - X_1)|}{s} \sum_{|M| \leq |J|} |(Z^M \Gamma)(X_1)|$$

and

$$|(Z^J \Gamma)(X_1)| \lesssim \sum_{|M| \leq |J|} |(Z^M \Gamma)(X_1)|$$

Proof. First note that $\left|\frac{\overline{Z}^I(X_1)}{s}\right| \lesssim 1$ for any I. Before continuing the proof we shall show a property on $\partial^k \Gamma$. There exists function A that are smooth when $|x|/t \leq c < 1$ such that

$$\partial^k \Gamma(t, x) = t^{-k} \sum_{|K| \leq k} A_K(x/t)(Z^K \Gamma)(t, x)$$

Let's prove this property by induction on k.
For $k = 1$ we have

$$\partial_{x^i} \Gamma = \frac{1}{t}[B_i \Gamma - x^i \partial_t \Gamma]$$

$$\partial_t \Gamma = \frac{1}{t}[S\Gamma - x^k \partial_{x^k} \Gamma]$$

The result follows from a straightforward induction.
Applying the lemma 6.2 we have

$$|\overline{Z}^J(\Gamma(X) - \Gamma(X_1))| \lesssim \sum_{k \leq |J|/2} s^k |(\partial^k \Gamma)(X) - (\partial^k \Gamma)(X_1)| + \sum_{k \leq |J|/2} \sum_{1 \leq l \leq k} C^{k-1} s^{k-1} |Z^l(X - X_1)| |\partial^k \Gamma(X)|$$

We have :

$$\partial^k \Gamma(t, x) = t^{-k} \sum_{|K| \leq k} A_K(x/t)(Z^K \Gamma)(t, x)$$

Thus

$$\sum_{k \leq |J|/2} \sum_{1 \leq l \leq k} C^{k-1} s^{k-1} |Z^l(X - X_1)| |\partial^k \Gamma(X)| \leq \sum_{|K| \leq |J|/2} |Z^K \Gamma(X)| \sum_{|M| \leq |J|} \frac{|\overline{Z}^M(X - X_1)|}{s}$$

Moreover

$$s^k |(\partial^k \Gamma)(X) - (\partial^k \Gamma)(X_1)| \lesssim \sum_{|K| \leq k} A_K(X/s)(Z^K \Gamma)(X) - A_K(X_1/s)(Z^K \Gamma)(X_1)$$

and

$$A_K(X/s)(Z^K\Gamma)(X) - A_K(X_1/s)(Z^K\Gamma)(X_1) = [(Z^K\Gamma)(X) - (Z^K\Gamma)(X_1)]A_K(X/s) + [A_K(X/s) - A_K(X_1/s)](Z^K\Gamma)(X_1)$$

Therefore

$$s^k|(\partial^k\Gamma)(X) - (\partial^k\Gamma)(X_1)| \lesssim \sum_{|K|\le k} |(Z^K\Gamma)(X) - (Z^K\Gamma)(X_1)| + \frac{|X-X_1|}{s}\sum_{|K|\le k}|Z^K\Gamma(X_1)|$$

and the result follows.

□

From the following Lemma we have :

Lemma 6.5. *Suppose $|Z^K X| \le Cs$ for $|K| \le |I|/2$. Then*

$$|\overline{Z}^J(\Gamma(X) - \Gamma(X_1))| \lesssim \sum_{|K|\le|J|/2+1} \|(Z^K\Gamma)(s,.)\|_\infty \sum_{|M|\le|J|} \frac{|\overline{Z}^M(X-X_1)|}{s} + \sum_{|M|\le|J|} |(Z^M\Gamma)(X) - (Z^M\Gamma)(X_1)|$$
$$+ \sum_{|K|\le|J|/2} \frac{|\overline{Z}^K(X-X_1)|}{s} \sum_{|M|\le|J|} |(Z^M\Gamma)(X_1)|$$

and

$$|(Z^J\Gamma)(X_1)| \lesssim \sum_{|M|\le|J|} |(Z^M\Gamma)(X_1)|$$

Before applying the vector fields to (6.17) we first decompose depending on whether more derivatives fall on Λ or Γ. We get

$$\overline{Z}^I(\hat\Gamma(X,\hat P) - \hat\Gamma(X_1,\hat P_1)) = \sum_{J+L=I,\ |J|\ge|I|/2,\ |L|\le|I|/2} \overline{Z}^J(\Gamma(X) - \Gamma(X_1))\cdot\overline{Z}^L\Lambda(\hat P) + \overline{Z}^J\Gamma(X_1)\cdot\overline{Z}^L(\Lambda(\hat P) - \Lambda(\hat P_1))$$
$$+ \sum_{J+L=I,\ |J|<|I|/2,\ |L|>|I|/2} \overline{Z}^J(\Gamma(X) - \Gamma(X_1))\cdot\overline{Z}^L\Lambda(\hat P_1) + \overline{Z}^J\Gamma(X)\cdot\overline{Z}^L(\Lambda(\hat P) - \Lambda(\hat P_1))$$

With this decomposition,lemma 6.3 and lemma 6.5 we get :

Proposition 6.6. *Suppose $|Z^K X|/s + |Z^K\hat P| \le C$ for $|K| \le |I|/2$. Then*

$$|\overline{Z}^I(\hat\Gamma(X,\hat P) - \hat\Gamma(X_1,\hat P_1))| \lesssim \sum_{|K|\le|I|/2+1} \|(Z^K\Gamma)(s,.)\|_\infty \sum_{|J|\le|I|} \Big(\frac{|\overline{Z}^J(X-X_1)|}{s} + |\overline{Z}^J(\hat P - \hat P_1)|\Big)$$
$$+ \sum_{|J|\le|I|} \Big(|(Z^J\Gamma)(X) - (Z^J\Gamma)(X_1)| + \sum_{|K|\le|I|/2} \Big(\frac{|\overline{Z}^K(X-X_1)|}{s} + |\overline{Z}^K(\hat P - \hat P_1)|\Big)|(Z^J\Gamma)(X_1)|\Big)$$

and

$$|Z^I(\hat\Gamma(X_1,P_1))| \lesssim \sum_{|J|\le|I|} |(Z^J\Gamma)(X_1))|$$

Corollary 6.2. Suppose $|Z^K X|/s + |Z^K\hat P| \le C$ for $|K| \le |K|/2$. Then

$$|\overline{Z}^I(\hat\Gamma(X,\hat P) - \hat\Gamma(X_1,\hat P_1))| \lesssim \sum_{|K|\le|I|+1} \|(Z^K\Gamma)(s,.)\|_\infty \sum_{|J|\le|I|} \Big(\frac{|\overline{Z}^J(X-X_1)|}{s} + |\overline{Z}^J(\hat P - \hat P_1)|\Big)$$

$$|Z^I(\hat\Gamma(X_1,P_1))| \lesssim \sum_{|J|\le|I|} \|(Z^J\Gamma)(s,.)\|_\infty$$

6.4 Parameter derivatives of the equations and vector fields applied to their differences

Since the vector fields applied to the system will be control by integrating from the final time t we need to control the final conditions of $\overline{Z}^I\overline{X}$ and $\overline{Z}^I\overline{P}$. In order to have some estimates on these final condition we need to estimate derivatives d/ds of our system. If we write $\hat{X}(s) = (s, X(s))$ we can write the following system

$$\frac{d\hat{X}^{(k)}}{ds} = \hat{P}^{(k)}, \qquad \frac{d\hat{P}^{(k)}}{ds} = \hat{\Gamma}^{(k)}(\hat{X}, \hat{P}), \quad \text{where} \quad \hat{X}^{(k)} = \frac{d^k\hat{X}}{ds}, \quad \hat{P}^{(k)} = \frac{d^k\hat{P}}{ds}$$

where

(6.21)
$$\hat{\Gamma}^{(k)}(\hat{X}, \hat{P}) = \Gamma^{(k)}(\hat{X})\cdot\Lambda^{(k)}(\hat{P}) := \sum_{m\geq 1,\, k_1+\ldots+k_m+m=k+1,\, 0\leq k_1\leq\ldots\leq k_m\leq k} (\partial^{k_1}\Gamma)(\hat{X})\cdots(\partial^{k_m}\Gamma)(\hat{X})\cdot\Lambda_{k_1,\ldots,k_m}(\hat{P})$$

The functions $\Lambda_{k_1,\ldots,k_m}(\hat{P})$ can be found by applying the Leibniz rule and the chain rule. For instance for $k = 1$ we have

$$\frac{d}{ds}(\Gamma(\hat{X})\cdot\Lambda(\hat{P})) = (\hat{P}\partial\Gamma(\hat{X}))\cdot\Lambda(\hat{P}) + \Gamma(\hat{X})\cdot(\partial\Lambda(\hat{P})(\Gamma(\hat{X})\cdot\Lambda(\hat{P})))$$

Now recall the system

$$\frac{d\hat{X}_2}{ds} = \hat{P}_2, \qquad \frac{d\hat{P}_2}{ds} = \Gamma(\hat{X}_1)\cdot\Lambda(\hat{P}_1), \quad \text{and} \quad \frac{d\hat{X}_1}{ds} = \hat{P}_1, \quad \frac{d\hat{P}_1}{ds} = 0.$$

From a straightforwars induction it can be shown that

$$\frac{dP_2^{(k)}}{ds} = \partial^k\Gamma(\hat{X}_1)\hat{P}_1^k\Lambda(\hat{P}_1)$$

Thus by writing

(6.22)
$$\hat{\Gamma}^{(k,2)} = \Gamma^{(k,2)}(\hat{X})\cdot\Lambda^{(k)}(\hat{P}) = \sum_{m\geq 2,\, k_1+\ldots+k_m+m=k+1,\, 0\leq k_1\leq\ldots\leq k_m\leq k} (\partial^{k_1}\Gamma)(\hat{X})\cdots(\partial^{k_m}\Gamma)(\hat{X})\cdot\Lambda_{k_1,\ldots,k_m}(\hat{P})$$

we have

(6.23)
$$\frac{d\overline{X}^{(k)}}{ds} = \overline{P}^{(k)}, \qquad \frac{d\overline{P}^{(k)}}{ds} = \hat{\Gamma}^{(k)}(\hat{X}, \hat{P}) - \hat{\Gamma}^{(k)}(\hat{X}_1, \hat{P}_1) + \hat{\Gamma}^{(k,2)}(\hat{X}_1, \hat{P}_1), \quad \text{where} \quad \overline{X}^{(k)} = \frac{d^k\overline{X}}{ds}, \quad \overline{P}^{(k)} = \frac{d^k\overline{P}}{ds}$$

Recall that in the proof of Lemma 6.4 we have shown that there exists function A that are smooth when $|x|/t \leq c < 1$ such that

$$\partial^k\Gamma(t,x) = t^{-k}\sum_{|K|\leq k} A_K(x/t)(Z^K\Gamma)(t,x)$$

We now have the following Lemma :

Lemma 6.6. *Suppose that $t|Z^I\Gamma(t,x)| \lesssim 1$ for $|I| \leq (|L|+k)/2$. Then*

$$t^k|Z^L\Gamma^{(k)}(t,x)| \lesssim \sum_{|I|\leq|L|+k} |Z^I\Gamma(t,x)|,$$

$$t^k|Z^L\Gamma^{(k,2)}(t,x)| \lesssim \sum_{|M|\leq(|L|+k)/2} t|Z^I\Gamma(t,x)| \sum_{|I|\leq|L|+k} |Z^I\Gamma(t,x)|,$$

Thanks to this lemma, we can now apply the estimates of the previous subsection to $\hat{\Gamma}^{(k)}$ instead of $\hat{\Gamma}$.

So as in Proposition 6.6 we obtain :

Proposition 6.7. *Suppose $|Z^KX|/s + |Z^K\hat{P}| \leq C$ for $|K| \leq |L|/2$. Set $n = |L| + k$. Then*

$$s^k|\overline{Z}^L(\hat\Gamma^{(k)}(X,\hat P)-\hat\Gamma^{(k)}(X_1,\hat P_1))| \lesssim \sum_{|K|\le n/2+1} \|(Z^K\Gamma)(s,.)\|_\infty \sum_{|J|\le|L|}\Big(\frac{|\overline{Z}^J(X-X_1)|}{s}+|\overline{Z}^J(\hat P-\hat P_1)|\Big)$$
$$+\sum_{|J|\le n}\Big(|(Z^J\Gamma)(X)-(Z^J\Gamma)(X_1)|+\sum_{|K|\le|L|/2}\Big(\frac{|\overline{Z}^K(X-X_1)|}{s}+|\overline{Z}^K(\hat P-\hat P_1)|\Big)|(Z^J\Gamma)(X_1)|\Big)$$

and

$$s^k|\overline{Z}^L(\hat\Gamma^{(k,2)}(X_1,\hat P_1))| \lesssim \sum_{|K|\le n/2} s\|(Z^K\Gamma)(s,.)\|_\infty^2+\sum_{|K|\le n/2} s\|(Z^K\Gamma)(s,.)\|_\infty\sum_{|M|\le n}|(Z^M\Gamma)(X_1)|$$

Corollary 6.3. Suppose $|Z^K X|/s+|Z^K\hat P|\le C$ for $|K|\le|L|/2$, and $|Z^M X_1|/s+|Z^M\hat P_1|\le C$ for $|M|\le|L|$. Set $n=|L|+k$. Then

$$s^k|\overline{Z}^L(\hat\Gamma^{(k)}(X,\hat P)-\hat\Gamma^{(k)}(X_1,\hat P_1))| \lesssim \sum_{|K|\le n+1} \|(Z^K\Gamma)(s,.)\|_\infty \sum_{|J|\le|L|}\Big(\frac{|\overline{Z}^J(X-X_1)|}{s}+|\overline{Z}^J(\hat P-\hat P_1)|\Big)$$

and

$$s^k|\overline{Z}^L(\hat\Gamma^{(k,2)}(X_1,\hat P_1))| \lesssim \sum_{|K|\le n} s\|(Z^K\Gamma)(s,.)\|_\infty^2$$

6.5 The final conditions

For $Y(s,t,x,\hat p)$ we have

(6.24)
$$\begin{aligned}\overline{Z}(Y(t,t,x,\hat p))| &= \overline{Z}(t)\frac{dY}{ds}(t,t,x,\hat p)+\overline{Z}(t)\frac{dY}{dt}(t,t,x,\hat p)\\ &= \overline{Z}(t)\frac{dY}{ds}(t,t,x,\hat p)+(\overline{Z}Y)(t,t,x,\hat p),\quad \text{where}\quad (\overline{Z}Y)(t,t,x,\hat p)=\overline{Z}(Y(s,t,x,\hat p))|_{s=t}\end{aligned}$$

Repeating (6.24) leads to
(6.25)
$$(\overline{Z}^IY)(t,t,x,\hat p)=\overline{Z}^I(Y(t,t,x,\hat p))+\sum_{J_1+\ldots+J_{k+1}+J=I,\ |J_i|\ge1,\ k\ge0} C_{I,J_1,\ldots,J_{k+1},J}\overline{Z}^{J_1}(t)\cdots\overline{Z}^{J_{k+1}}(t)(\overline{Z}^JY^{(k+1)})(t,t,x,\hat p),$$

where $Y^{(k)}=\frac{d^kY}{ds}$ and $\overline{Z}^{J_i}(t)=Z^{J_i}(t)$ are constants time t or x^j for some j. This equality is a straightforward induction.
Applying this equality to $\overline{P}$ one can easily obtain

$$(\overline{Z}^I\overline{P})(t,t,x,\hat p)=\sum_{J_1+\ldots+J_{k+1}+J=I,\ |J_i|\ge1,\ k\ge0} C_{I,J_1,\ldots,J_{k+1},J}\overline{Z}^{J_1}(t)\cdots\overline{Z}^{J_{k+1}}(t)(\overline{Z}^J\overline{P}^{(k+1)})(t,t,x,\hat p),$$

From the previous equation we also have

$$(\overline{Z}^I\overline{X})(t,t,x,\hat p)=\sum_{J_1+\ldots+\underline{J_{k+2}}+J=I,\ |J_i|\ge1,\ k\ge0} C_{I,J_1,\ldots,\underline{J_{k+2}},J}\overline{Z}^{J_1}(t)\cdots\overline{Z}^{J_{k+1}}(t)(\overline{Z}^J\overline{P}^{(k+1)})(t,t,x,\hat p),$$

where the fact that $\overline{X}(t,t,x,\hat p)=\overline{P}(t,t,x,\hat p)=0$ has been used.

Recall that in this subsection we want to control the final conditions of $\overline{Z}^I\overline{X}$ and $\overline{Z}^I\overline{P}$ at final time t. Using the decomposition (6.23) and the following inequalities :

$$|(\overline{Z}^I\overline{P})(t,t,x,\hat p)| \lesssim \sum_{|L|+k\le|I|-1,\ k\ge0} t^{k+1}|(\overline{Z}^L\overline{P}^{(k+1)})(t,t,x,\hat p)|$$
$$|(\overline{Z}^I\overline{X})(t,t,x,\hat p)| \lesssim \sum_{|L|+k\le|I|-2,\ k\ge0} t^{k+2}|(\overline{Z}^L\overline{P}^{(k+1)})(t,t,x,\hat p)|$$

we have

Lemma 6.7.

$$|(\overline{Z}^I\overline{P})| \lesssim \sum_{|L|+k\leq|I|-1,\, k\geq 0} t^{k+1}|\overline{Z}^L(\hat{\Gamma}^{(k)}(\hat{X},\hat{P})-\hat{\Gamma}^{(k)}(\hat{X}_1,\hat{P}_1))|+t^{k+1}|\overline{Z}^L(\hat{\Gamma}^{(k,2)}(\hat{X}_1,\hat{P}_1))| \qquad at \quad s=t$$

$$|(\overline{Z}^I\overline{X})| \lesssim \sum_{|L|+k\leq|I|-2,\, k\geq 0} t^{k+2}|\overline{Z}^L(\hat{\Gamma}^{(k)}(\hat{X},\hat{P})-\hat{\Gamma}^{(k)}(\hat{X}_1,\hat{P}_1))|+t^{k+2}|\overline{Z}^L(\hat{\Gamma}^{(k,2)}(\hat{X}_1,\hat{P}_1))| \qquad at \quad s=t$$

where everything is evaluated at $(s,t,x,\hat{p})$ where $s=t$.

The previous estimates lead us to formulate the following proposition :

Proposition 6.8. *Suppose $t\geq t_0+1$, $|x|\leq ct$, $(t,x,\hat{p})\in supp(f)$ and the bound (6.1) hold. Then, for $|I|\leq\lfloor\frac{N}{2}\rfloor+2$,*

$$|\overline{Z}^I(\overline{P}^i(s,t,x,\hat{p})|_{s=t}+t^{-1}|\overline{Z}^I(\overline{X}^i(s,t,x,\hat{p})|_{s=t}\lesssim\frac{\epsilon}{t^{2a}}$$

Proof. We shall prove this by induction. The case $|I|=0$ is straightforward. Assuming its true for $|I|<m\leq N/2+1$ the assumptions of corollary 6.13 hold at $s=t$ and we can use it.

$$\begin{aligned}t^k|\overline{Z}^L(\hat{\Gamma}^{(k)}(X,\hat{P})-\hat{\Gamma}^{(k)}(X_1,\hat{P}_1))| &\lesssim \sum_{|K|\leq|L|+k+1}\|(Z^K\Gamma)(s,.)\|_\infty\sum_{|J|\leq|L|}\Big(\frac{|\overline{Z}^J(X-X_1)|}{t}+|\overline{Z}^J(\hat{P}-\hat{P}_1)|\Big)\\ &\lesssim\frac{\epsilon}{t^{1+a}}\sum_{|J|\leq|L|}\Big(\frac{|\overline{Z}^J(X-X_1)|}{t}+|\overline{Z}^J(\hat{P}-\hat{P}_1)|\Big)\end{aligned}$$

Thus

$$t^{k+1}|\overline{Z}^L(\hat{\Gamma}^{(k)}(X,\hat{P})-\hat{\Gamma}^{(k)}(X_1,\hat{P}_1))|\lesssim\frac{\epsilon}{t^a}\sum_{|J|\leq|L|}\Big(\frac{|\overline{Z}^J(X-X_1)|}{t}+|\overline{Z}^J(\hat{P}-\hat{P}_1)|\Big)$$

Writing $X-X_1=\overline{X}+X_2-X_1$ and $\hat{P}-\hat{P}_1=\overline{P}+\hat{P}_2-\hat{P}_1$ we have

$$t^{k+1}|\overline{Z}^L(\hat{\Gamma}^{(k)}(X,\hat{P})-\hat{\Gamma}^{(k)}(X_1,\hat{P}_1))|\lesssim\frac{\epsilon}{t^{2a}}+\frac{\epsilon}{t^a}\sum_{|J|\leq|L|}\Big(\frac{|\overline{Z}^J\overline{X}|}{s}+|\overline{Z}^J\overline{P}|\Big)$$

$$t^{k+1}|\overline{Z}^L(\hat{\Gamma}^{(k,2)}(X_1,\hat{P}_1))|\lesssim\frac{\epsilon}{t^{2a}}$$

where the estimates (6.14) and (6.16) from Corollary 6.1 have been used. We then use the previous lemma which concludes the induction.

□

Proposition 6.9. *Suppose $t\geq t_0+1$, $|x|\leq ct$, $(t,x,\hat{p})\in supp(f)$ and the bound (6.1) hold. Then, for $|I|\leq N$,*

$$\Big|\overline{Z}^J(\overline{P}^i(s,t,x,\hat{p})|_{s=t}\Big|\lesssim\frac{\epsilon}{t^{2a}}+\frac{1}{t^a}\sum_{|J|\leq|I|-1}\Big(t|(Z^J\Gamma)(t,x)|+\int_{t_0}^t\frac{s'-t_0}{t-t_0}\Big|(Z^J\Gamma)(s',s'\frac{x}{t})\Big|ds'\Big)$$

$$\Big|\overline{Z}^J(\overline{X}^i(s,t,x,\hat{p})|_{s=t}\Big|\lesssim\frac{\epsilon}{t^{2a-1}}+t^{1-a}\sum_{|J|\leq|I|-1}\Big(t|(Z^J\Gamma)(t,x)|+\int_{t_0}^t\frac{s'-t_0}{t-t_0}\Big|(Z^J\Gamma)(s',s'\frac{x}{t})\Big|ds'\Big)$$

Proof. As in the previous proof, we use an induction. The case $|I|=0$ is straightforward. Assuming its true for $|I|<m\leq N/2+1$ w can use proposition 6.7 at $s=t$:

Thus

$$\begin{aligned}t^k|\overline{Z}^L(\hat{\Gamma}^{(k)}(X,\hat{P})-\hat{\Gamma}^{(k)}(X_1,\hat{P}_1))|\lesssim\frac{\epsilon}{t^{1+a}}\sum_{|J|\leq|L|}\Big(\frac{|\overline{Z}^J(X-X_1)|}{s}+|\overline{Z}^J(\hat{P}-\hat{P}_1)|\Big)+\\ \frac{\epsilon}{t^{1+a}}\sum_{|J|\leq|L|}\frac{\epsilon}{t^a}+\sum_{|J|\leq|L|+k}t|(Z^J\Gamma)(X_1)|\end{aligned}$$

Thus

$$t^{k+1}|\overline{Z}^L(\hat{\Gamma}^{(k)}(X,\hat{P})-\hat{\Gamma}^{(k)}(X_1,\hat{P}_1))| \lesssim \frac{1}{t^a}\sum_{|J|\leq|L|}\Big(\epsilon\frac{|\overline{Z}^J(X-X_1)|}{s}+\epsilon|\overline{Z}^J(\hat{P}-\hat{P}_1)|+\frac{\epsilon}{t^a}+\sum_{|J|\leq|L|+k} t|(Z^J\Gamma)(X_1)|\Big)$$

and

$$t^{k+1}|\overline{Z}^L(\hat{\Gamma}^{(k,2)}(X_1,\hat{P}_1))| \lesssim \frac{\epsilon}{t^{2a}}+\frac{1}{t^a}\sum_{|M|\leq|L|+k} t|(Z^M\Gamma)(X_1)|$$

Writing $X-X_1=\overline{X}+X_2-X_1$ and $\hat{P}-\hat{P}_1=\overline{P}+\hat{P}_2-\hat{P}_1$ and using the estimates (6.13) and (6.15) the same way we used estimates in the previous proof the induction is verified. Then, the proposition follows.

□

6.6 L^∞ estimates for lower order derivatives of geodesics

Proposition 6.10. *Suppose $t\geq t_0+1$, $|x|\leq ct$, $(t,x,\hat{p})\in supp(f)$ and the bound (6.1) hold. Then, for $|I|\leq \lfloor\frac{N}{2}\rfloor+2$ and $i=1,2,3$,*

$$s^{2a-1}\Big|\overline{Z}^I\big(\overline{X}^i(s,t,x,p))\big)\Big|+s^{2a}\Big|\overline{Z}^I\big(\overline{P}^i(s,t,x,p))\big)\Big|\leq C\epsilon$$

for all $t_0\leq s\leq t$

Proof. We shall proceed by induction. The case $|I|=0$ is straightforward. Assuming the result is true for $|I|\leq k$ then the assumptions of Corollary 6.2 are true. We obtain

$$|\overline{Z}^I(\hat{\Gamma}(X,\hat{P})-\hat{\Gamma}(X_1,\hat{P}_1))| \lesssim \sum_{|K|\leq|I|+1}\|(Z^K\Gamma)(s,.)\|_\infty\sum_{|J|\leq|I|}\Big(\frac{|\overline{Z}^J(X-X_1)|}{s}+|\overline{Z}^J(\hat{P}-\hat{P}_1)|\Big)$$

Hence

$$\Big|\frac{d}{ds}\overline{Z}^I(\overline{P}^i(s))\Big| = |\overline{Z}^I(\hat{\Gamma}(X,\hat{P})-\hat{\Gamma}(X_1,\hat{P}_1))| \lesssim \frac{\epsilon}{s^{1+a}}\sum_{|J|\leq|I|}\Big(\frac{|\overline{Z}^J(X-X_1)|}{s}+|\overline{Z}^J(\hat{P}-\hat{P}_1)|\Big)$$

Using the inequalities (6.14) and (6.16) after writing $X-X_1=\overline{X}+X_2-X_1$ and $\hat{P}-\hat{P}_1=\overline{P}+\hat{P}_2-\hat{P}_1$ leads us to the bound

$$\Big|\frac{d}{ds}\overline{Z}^I(\overline{P}^i(s))\Big| =\lesssim \frac{\epsilon}{s^{1+2a}}+\frac{\epsilon}{s^{1+a}}\sum_{|J|\leq|I|}\Big(\frac{|\overline{Z}^J(\overline{X}(s))|}{s}+|\overline{Z}^J(\overline{P}(s))|\Big)$$

We now need to integrate between s and t and thus the bounds found in the previous section are used here. So, using proposition 6.8 in order to recover a bound on $|\overline{Z}^I\big(\overline{P}^i(s,t,x,p))\big)\Big|$ at $s=t$ we have

$$\Big|\overline{Z}^I(\overline{P}^i(s))\Big| \lesssim \frac{\epsilon}{s^{2a}}+\epsilon\sum_{|J|\leq|I|}\int_s^t\Big(\frac{|\overline{Z}^J(\overline{X}(s'))|}{s'^{2+a}}+\frac{|\overline{Z}^J(\overline{P}(s'))|}{s'^{1+a}}\Big)ds',$$

The Grönwall inequality gives us

$$\Big|\overline{Z}^I(\overline{P}^i(s))\Big| \lesssim \frac{\epsilon}{s^{2a}}+\epsilon\sum_{|J|\leq|I|}\int_s^t\Big(\frac{|\overline{Z}^J(\overline{X}(s'))|}{s'^{2+a}}\Big)ds',$$

Integrating again between s and t one gets

$$\Big|\overline{Z}^I(\overline{X}^i(s))\Big| \lesssim \epsilon+\epsilon\sum_{|J|\leq|I|}\int_s^t\Big(\frac{|\overline{Z}^J(\overline{X}(s'))|}{s'^{1+a}}\Big)ds',$$

where the proposition 6.8 has also been used.

Using once again the Grönwall inequality gives the result.

□

Proposition 6.11. *Suppose $t \geq t_0+1$, $|x| \leq ct$, $(t,x,\hat{p}) \in supp(f)$ and the bound (6.1) hold. Then, for $|I| \leq \lfloor \frac{N}{2} \rfloor + 2$ and $i = 1,2,3$,*

$$s^{-1}\left|\overline{Z}^I\left(X^i(s,t,x,p)\right)\right| + \left|\overline{Z}^I\left(\hat{P}^i(s,t,x,p)\right)\right| \leq C\epsilon$$

for all $t_0 \leq s \leq t$

Proof. The previous proposition gives us

$$s^{2a-1}\left|\overline{Z}^I\left(\overline{X}^i(s,t,x,p)\right)\right| + s^{2a}\left|\overline{Z}^I\left(\overline{P}^i(s,t,x,p)\right)\right| \leq C\epsilon$$

Now we use $X = \overline{X} + X_2 - X_1 + X_1$ and $\hat{P} = \overline{P} + \hat{P}_2 - \hat{P}_1 + \hat{P}_1$ The estimates on $\frac{\overline{Z}^I\overline{X}}{s}$ and $\overline{Z}^I\overline{P}$ come from the previous inequality. The estimates on $\frac{\overline{Z}^I(X_2-X_1)}{s}$ and $\overline{Z}^I(\hat{P}_2 - \hat{P}_1)$ come from Corollary 6.1 Finally the estimates on $\frac{\overline{Z}^I X_1}{s}$ and $\overline{Z}^I\hat{P}_1$ come from Proposition 6.5 which concludes the proof. □

In the next subsection we provide the main result of this section which is a high order estimate on the vector field applied to the geodesics.

6.7 Higher order estimates for derivatives of geodesics

Proposition 6.12. *Suppose $t \geq t_0+1$, $|x| \leq ct$, $(t,x,\hat{p}) \in supp(f)$ and the bound (6.1) hold. Then, for $|I| \leq N$ and $i = 1,2,3$,*

$$\left|\overline{Z}^I(\overline{X}^i(s,t,x,\hat{p}))\right| \lesssim \frac{\epsilon}{s^{2a-1}} + t^{1-a}\sum_{|J|\leq|I|-1} t|(Z^J\Gamma)(t,x)| + \sum_{|J|\leq|I|}\int_{t_0}^t (s')^{1-a}\left|(Z^J\Gamma)(s',s'\frac{x}{t})\right|ds' \tag{6.26}$$

$$\begin{aligned}\left|\overline{Z}^I(\overline{P}^i(s,t,x,\hat{p}))\right| \lesssim{}& \frac{\epsilon}{s^{2a}} + \frac{1}{s^a}\sum_{|J|\leq|I|-1} t|(Z^J\Gamma)(t,x)| + \sum_{|J|\leq|I|}\int_{t_0}^t \frac{1}{s^a}\left|(Z^J\Gamma)(s',s'\frac{x}{t})\right|ds' \\ &+ \sum_{|J|\leq|I|}\int_s^t \left|(Z^J\Gamma)(s',s'\frac{x}{t}) - (Z^J\Gamma)(s',X(s'))ds'\right|\end{aligned} \tag{6.27}$$

for all $t_0 \leq s \leq t$, $|I| \leq N$

Proof. From proposition 6.6 we have

$$\begin{aligned}\left|\frac{d}{ds}\overline{Z}^I(\overline{P}^i(s))\right| \lesssim{}& \sum_{|K|\leq|I|/2+1}\|(Z^K\Gamma)(s,.)\|_\infty \sum_{|J|\leq|I|}\Big(\frac{|\overline{Z}^J(X-X_1)|}{s} + |\overline{Z}^J(\hat{P}-\hat{P}_1)|\Big) \\ &+ \sum_{|J|\leq|I|}\Big(|(Z^J\Gamma)(X) - (Z^J\Gamma)(X_1)| + \sum_{|K|\leq|I|/2}\Big(\frac{|\overline{Z}^K(X-X_1)|}{s} + |\overline{Z}^K(\hat{P}-\hat{P}_1)|\Big)|(Z^J\Gamma)(X_1)|\Big)\end{aligned}$$

As in the previous proof we use $X - X_1 = \overline{X} + X_2 - X_1$ and $\hat{P} - \hat{P}_1 = \overline{P} + \hat{P}_2 - \hat{P}_1$ and with proposition 6.10 and corollary 1 and the bound (6.1) we obtain

$$\begin{aligned}\left|\frac{d}{ds}\overline{Z}^I(\overline{P}^i(s))\right| \lesssim{}& \frac{\epsilon}{s^{1+a}}\sum_{|J|\leq|I|}\Big(\frac{|\overline{Z}^J(\overline{X}(s))|}{s} + |\overline{Z}^J(\overline{P}(s))|\Big) + \frac{\epsilon}{s^{1+2a}} \\ &+ \sum_{|J|\leq|I|}|(Z^J\Gamma)(s,X(s)) - (Z^J\Gamma)(s,s\frac{x}{t})| + \frac{t}{s^{1+a}}\sum_{|J|\leq|I|-1}\left|(Z^J\Gamma(t,x))\right| + \frac{1}{s^a}\sum_{|J|\leq|I|}\left|(Z^J\Gamma(s,s\frac{x}{t}))\right| \\ &+ \frac{1}{s^{1+a}}\sum_{|J|\leq|I|}\int_{t_0}^t\left|(Z^J\Gamma(s',s'\frac{x}{t}))\right|\frac{s'}{s'+s}ds'\end{aligned}$$

We call $F_I(s,t,x,\hat{p})$:

$$F_I(s,t,x,\hat{p}) = \frac{\epsilon}{s^{1+2a}} + \sum_{|J|\leq |I|} |(Z^J\Gamma)(s,X(s)) - (Z^J\Gamma)(s,s\frac{x}{t})| + \frac{t}{s^{1+a}} \sum_{|J|\leq |I|-1} \Big|(Z^J\Gamma(t,x))\Big| + \frac{1}{s^a}\sum_{|J|\leq |I|}\Big|(Z^J\Gamma(s,s\frac{x}{t}))\Big|$$
$$+ \frac{1}{s^{1+a}}\sum_{|J|\leq|I|}\int_{t_0}^{t}\Big|(Z^J\Gamma(s',s'\frac{x}{t}))\Big|\frac{s'}{s'+s}ds'$$

Integrating between s and t we obtain,

$$\left|\overline{Z}^I(\overline{P}^i(s))\right| \lesssim \left|\overline{Z}^I(\overline{P}^i(s))\right|_{s=t} + \epsilon\int_s^t \sum_{|J|\leq |I|}\Big(\frac{|\overline{Z}^J(\overline{X}(\tilde{s}))|}{\tilde{s}^{2+a}} + \frac{|\overline{Z}^J(\overline{P}(\tilde{s}))|}{\tilde{s}^{1+a}}\Big) + F_I(\tilde{s},t,x,\hat{p})d\tilde{s}$$

The Grönwall inequality gives

$$\left|\overline{Z}^I(\overline{P}^i(s))\right| \lesssim \left|\overline{Z}^I(\overline{P}^i(s))\right|_{s=t} + \epsilon\int_s^t \sum_{|J|\leq |I|}\frac{|\overline{Z}^J(\overline{X}(\tilde{s}))|}{\tilde{s}^{2+a}} + F_I(\tilde{s},t,x,\hat{p})d\tilde{s}$$

And integrating again between s and t we obtain

$$\left|\overline{Z}^I(\overline{X}^i(s))\right| \lesssim \left|\overline{Z}^I(\overline{X}^i(s))\right|_{s=t} + (t-s)\left|\overline{Z}^I(\overline{P}^i(s))\right|_{s=t} + \epsilon\int_s^t \sum_{|J|\leq |I|}\frac{|\overline{Z}^J(\overline{X}(\tilde{s}))|}{\tilde{s}^{1+a}} + \tilde{s}F_I(\tilde{s},t,x,\hat{p})d\tilde{s}$$

Using again the Grönwall inequality :

$$\left|\overline{Z}^I(\overline{X}^i(s))\right| \lesssim \left|\overline{Z}^I(\overline{X}^i(s))\right|_{s=t} + (t-s)\left|\overline{Z}^I(\overline{P}^i(s))\right|_{s=t} + \int_s^t \tilde{s}F_I(\tilde{s},t,x,\hat{p})d\tilde{s}$$

From proposition 6.9 we have

$$\left|\overline{Z}^J(\overline{P}^i(s,t,x,\hat{p})|_{s=t}\right| \lesssim \frac{\epsilon}{t^{2a}} + \frac{1}{t^a}\sum_{|J|\leq|I|-1}\Big(t|(Z^J\Gamma)(t,x)| + \int_{t_0}^t \frac{s'-t_0}{t-t_0}\Big|(Z^J\Gamma)(s',s'\frac{x}{t})\Big|ds'\Big)$$

$$\left|\overline{Z}^J(\overline{X}^i(s,t,x,\hat{p})|_{s=t}\right| \lesssim \frac{\epsilon}{t^{2a-1}} + t^{1-a}\sum_{|J|\leq|I|-1}\Big(t|(Z^J\Gamma)(t,x)| + \int_{t_0}^t \frac{s'-t_0}{t-t_0}\Big|(Z^J\Gamma)(s',s'\frac{x}{t})\Big|ds'\Big)$$

Therefore we need to estimate $\int_s^t \tilde{s}F_I(\tilde{s},t,x,\hat{p})d\tilde{s}$ to have bound (6.20) and more precisely it remains to estimate

$$\int_s^t \frac{1}{\tilde{s}^a}\sum_{|J|\leq|I|}\int_{t_0}^t\Big|(Z^J\Gamma(\tilde{s},\tilde{s}\frac{x}{t}))\Big|\frac{s'}{s'+\tilde{s}}ds'd\tilde{s}$$

We have

$$\int_s^t \frac{1}{\tilde{s}^a}\int_{t_0}^t\Big|(Z^J\Gamma(\tilde{s},\tilde{s}\frac{x}{t}))\Big|\frac{s'}{s'+\tilde{s}}ds'd\tilde{s} \lesssim \int_s^t \frac{1}{\tilde{s}^a}\int_{\tilde{s}}^t \Big|(Z^J\Gamma(s',s'\frac{x}{t}))\Big|ds' + \frac{1}{\tilde{s}^{1+a}}\int_{t_0}^{\tilde{s}} s'\Big|(Z^J\Gamma(s',s'\frac{x}{t}))\Big|ds'd\tilde{s}$$

and the bound (4.30) follows. The bound (4.31) can be shown in a similar way.

□

From this proposition and proposition 6.5 we have the following Corollary :

Corollary 6.4. Suppose $t\geq t_0+1$, $|x|\leq ct$, $(t,x,\hat{p})\in \mathrm{supp}(f)$ and the bound (6.1) hold. Then, for $|I|\leq N$ and $i=1,2,3$,

$$\left|\overline{Z}^I(X^i(t_0,t,x,\hat{p})\right| + \left|\overline{Z}^I(\hat{P}^i(t_0,t,x,\hat{p})\right| \leq C\Big(1 + t^{2-a}\sum_{|J|\leq|I|-1}|(Z^J\Gamma)(t,x)|$$
$$+ \sum_{|J|\leq|I|}\int_{t_0}^t (s')^{1-a}\Big|(Z^J\Gamma)(s',s'\frac{x}{t})\Big| + s'\Big|(Z^J\Gamma)(s',s'\frac{x}{t}) - (Z^J\Gamma)(s',X(s'))\Big|ds'\Big)$$

Now that we've shown some results for the vector fields applied to geodesics in the case $t\geq t_0+1$ we shall have bounds when the time t is close to t_0, where the vector fields become singular and we don't need the approximations of geodesics sicne the result are local. W will note $\partial = (\partial_t, \partial_{x^1}, \partial_{x^2}, \partial_{x^3})$. We give some properties that can be shown by following the proof of this subsection. The proof of the next section are in fact much simpler than the previous ones.

6.8 Spacetime derivatives and small time

First, we have the following proposition which can easily be shown following the proof of proposition 6.6.

Proposition 6.13. *Let I be a multi index and suppose that $|\partial^K X|/s + |\partial^K \hat{P}| \leq C$ for all $|K| \leq \frac{|I|}{2}$. Then*

$$|\partial^I(\hat{\Gamma}(s, X(s), \hat{P}(s)))| \lesssim \sum_{|K| \leq |I|/2+1} \|(\partial^K \Gamma)(s, .)\|_\infty \sum_{|J| \leq |I|} \Big(|\partial^J X(s)| + |\partial^J \hat{P}(s)|\Big) + \sum_{|J| \leq |I|} |\partial^J \Gamma(s, X(s))|$$

The following propositions can also be shown with the same idea that in the previous subsection :

Proposition 6.14. *Let I be a multi index and suppose that $|\partial^J \Gamma(t, x)| \leq C$ for all $|J| \leq \left\lfloor \frac{|I|}{2} \right\rfloor + 1$. Then*

$$|(\partial^I \hat{P})(t, t, x, \hat{p})| \lesssim \sum_{|J| \leq |I|-1} |(\partial^J \Gamma)(t, x)|, \qquad |(\partial^I X)(t, t, x, \hat{p})| \lesssim 1 + \sum_{|J| \leq |I|-1} |(\partial^J \Gamma)(t, x)|$$

Proposition 6.15. *Suppose $t_0 \leq t \leq t_0 + 1$ and $|\partial^J \Gamma(t, x)| \leq C$ for all $t_0 \leq t' \leq t$ and $|x| \leq ct'$ and $|J| \leq \left\lfloor \frac{N}{2} \right\rfloor + 2$. Then, for $|I| \leq \left\lfloor \frac{N}{2} \right\rfloor + 2$,*

$$|(\partial^I X)(t, t, x, \hat{p})| + |(\partial^I \hat{P})(t, t, x, \hat{p})| \leq C,$$

for all $t_0 \leq s \leq t$.

Proposition 6.16. *Suppose $t_0 \leq t \leq t_0 + 1$ and $|\partial^J \Gamma(t, x)| \leq C$ for all $t_0 \leq t' \leq t$ and $|x| \leq ct'$ and $|J| \leq \left\lfloor \frac{N}{2} \right\rfloor + 2$. Then, for $|I| \leq N$,*

$$|(\partial^I X)(t, t, x, \hat{p})| + |(\partial^I \hat{P})(t, t, x, \hat{p})| \lesssim 1 + \sum_{|J| \leq |I|-1} |\partial^J \Gamma(t, x)| + \sum_{|J| \leq |I|} \int_s t |\partial^J \Gamma(s', X(s'))| ds',$$

for all $t_0 \leq s \leq t$.

Now we have all the ingredients to begin estimating the components of the stress-energy tensor. To understand how we can apply the result of the previous section on the stress-energy tensor it is interesting to understand it on a simpler case. The best way to understand it is to assimilate the paper written by Claude Bardos and Pierre Degond in 1985 on the Vlasov-Poisson system.

7 The Vlasov-Poisson system

In their paper [2], Claude Bardos and Pierre Degond show the dispersion property and the existence of a smooth solution for all time providing the initial data are localised and smooth enough.

7.1 The system

We denote by u_α, $(1 \leq \alpha \leq N)$, N positive functions which describe the density at the point x, and at the time t, of particles of the type α which have the velocity v. We denote by q_α and m_α their masse and charge. The variation of u_α is described by

$$\frac{\partial u_\alpha}{\partial t} + (v \cdot \nabla_x) u_\alpha + \frac{q_\alpha}{m_\alpha}(E \cdot \nabla_v) u_\alpha = 0 \tag{7.1}$$

and

$$E = 4\pi \sum_\alpha q_\alpha \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \Big(\int_{\mathbb{R}^3} u_\alpha(x, v, t) dv \Big) dy \tag{7.2}$$

We also write

$$\rho(t, x) = \sum_\alpha q_\alpha \int_{\mathbb{R}^3} u_\alpha(x, v, t) dv \tag{7.3}$$

If the functions $u_\alpha(x,v,t)$ are smooth then

$$t \to u_\alpha(x(t),v(t),t)$$

is constant if $(x(t),v(t))$ verify the following Hamiltonian system :

$$\dot{x}(t) = v(t), \qquad \dot{v}(t) = \frac{q_\alpha}{m_\alpha} E(x(t),t) \tag{7.4}$$

We now consider the Hamiltonian system :

$$\dot{X}(s) = V(s), \qquad \dot{V}(s) = E(X(s),s). \tag{7.5}$$

with the Cauchy data :

$$X(t) = x, \qquad V(t) = v \tag{7.6}$$

We can already see some similarities between the Vlasov-Poisson and the Vlasov-Einstein system. The function ρ could be assimilated to our stress-energy tensor in the case of Vlasov-Einstein system. We will see that such as in theorem 3.2, bounds on ρ are the ingredients to conclude.

7.2 Global existence of the solutions for small initial data

The following theorem is the core of the paper from Bardos and Degond, it gives the existence of a smooth solution for all time providing the initial data are localised and smooth enough. In the following subsection we do not show the entire proof of the global existence of the solutions. We will show an inequality that look like the same as our theorem 3.2 and we will stop until this inequality is shown.

Theorem 7.1. *We assume that the functions $u_{0\alpha}(x,v)$ for $\alpha = 1, ..., N$, are twice continuously differentiable, and that they satisfy the following estimates, for any pair (x,v) in $\mathbb{R}^3_x \times \mathbb{R}^3_v$, and any α :*

$$0 \le u_{0,\alpha}(x,v) \le \frac{\epsilon}{(1+|x|)^4(1+|v|)^4} \tag{7.7}$$

$$|\nabla_{(x,v)} u_{0,\alpha}(x,v)| \le \frac{\epsilon}{(1+|x|)^4(1+|v|)^5} \tag{7.8}$$

$$|\nabla^2_{(x,v)} u_{0,\alpha}(x,v)| \le \frac{\epsilon}{(1+|x|)^4(1+|v|)^4} \tag{7.9}$$

Then, there exists ϵ_0, such that for $\epsilon < \epsilon_0$ the Vlasov-Poisson equations

$$\frac{\partial u_\alpha}{\partial t} + (v \cdot \nabla_x) u_\alpha + \frac{q_\alpha}{m_\alpha}(E \cdot \nabla_v) u_\alpha = 0 \tag{7.10}$$

$$u_\alpha(x,v,0) = u_{0,\alpha}(x,v) \tag{7.11}$$

$$\rho(t,x) = \sum_\alpha q_\alpha \int_{\mathbb{R}^3} u_\alpha(x,v,t) dv \tag{7.12}$$

$$E = 4\pi \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho(y,t) dy \tag{7.13}$$

have a global in time solution such that $u_\alpha(x,v,t)$ is continuously differentiable in $\mathbb{R}^3_x \times \mathbb{R}^3_v \times \mathbb{R}^+_t$ and such that $E(x,t)$ and $\rho(x,t)$ are continuous in $\mathbb{R}^3_x \times \mathbb{R}^+_t$. Furthemore, this solution satisfies the following uniform estimates.

$$\|(u_\alpha(.,.,t))\|_\infty \le \epsilon, \qquad \|(u_\alpha(.,.,t))\|_1 \le C\epsilon \tag{7.14}$$

$$Sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} u_\alpha(x,v,t) dv \le \frac{C\epsilon}{(1+t)^3} \tag{7.15}$$

$$\int\int |\nabla_v u_\alpha(x,v,t)| dx dv \le C\epsilon t \tag{7.16}$$

$$Sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} |\nabla_v u_\alpha(x,v,t)| dv \le \frac{C\epsilon}{(1+t)^2} \tag{7.17}$$

$$\|E(.,t)\|_\infty \le \frac{C}{(1+t)^2} \tag{7.18}$$

$$\|\nabla_x E(.,t)\|_\infty + \|\nabla^2_x E(.,t)\|_\infty \le \frac{C}{(1+t)^{5/2}} \tag{7.19}$$

We notice that this theorem isn't far from our theorem 3.1 : we need some assumptions on the initial datas and we have decay estimate.

The idea of the proof is to proceed by iteration and introduce a sequence of functions

$$u^n(x,v) = (u_1^n(x,v), ..., u_\alpha^n(x,v), ..., u_N^n(x,v))$$

in the following manner : We first define $u_\alpha^1(x,v,t)$ by the formula :

$$\frac{\partial u_\alpha^1}{\partial t} + v \cdot \nabla_x u_\alpha^1 = 0 \qquad u_\alpha^1(x,v,0) = u_{0,\alpha}(x,v) \tag{7.20}$$

which is the linearized equation. and the charge ρ_1 and the Electric field E_1 are given by the equations :

$$\rho_1(t,x) = \sum_\alpha q_\alpha \int_{\mathbb{R}^3} u_\alpha^1(x,v,t)dv \tag{7.21}$$

$$E_1 = 4\pi \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho_1(y,t)dy \tag{7.22}$$

Now suppose that $u_\alpha^n(x,v,t)$ is defined. Then ρ_n and E_n are defined by the equations

$$\rho_n(t,x) = \sum_\alpha q_\alpha \int_{\mathbb{R}^3} u_\alpha^n(x,v,t)dv \tag{7.23}$$

$$E_n = 4\pi \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho_n(y,t)dy \tag{7.24}$$

and u_α^{n+1} is supposed to satisfy the linear transport equation

$$\frac{\partial u_\alpha^{n+1}}{\partial t} + (v \cdot \nabla_x) u_\alpha^{n+1} + \frac{q_\alpha}{m_\alpha}(E_n \cdot \nabla_v) u_\alpha^{n+1} = 0 \tag{7.25}$$

$$u_\alpha^{n+1}(x,v,0) = u_{\alpha,0}(x,v) \tag{7.26}$$

So the solution of the system is

$$u_\alpha^{n+1}(x,v,t) = u_{\alpha,0}(X_\alpha^n(0,t,x,v), V_\alpha^n(0,t,x,v)) \tag{7.27}$$

where $X_\alpha^n(0,t,x,v), V_\alpha^n(0,t,x,v)$ is the solution of the Hamiltonian system :

$$\dot{X}_\alpha^n(s) = V_\alpha^n(s), \qquad \dot{V}_\alpha^n(s) = E_n(X_\alpha^n(s), s). \tag{7.28}$$

with

$$X_\alpha^n(t) = x, \qquad V_\alpha^n(t) = v \tag{7.29}$$

In the sequel, X_α^n, V_α^n will mean $X_\alpha^n(0,t,x,v)$ and $V_\alpha^n(0,t,x,v)$ and C will denote a numerical constant, which may eventually depend on q_α and m_α only. We can also deduce the following uniform estimates :

$$0 \leq u_\alpha^{n+1}(x,v,t) \leq \epsilon, \qquad \int_{\mathbb{R}^3} |\rho_{n+1}(x,t)| dx \leq A\epsilon \tag{7.30}$$

In order to prove the convergence of the subsequences u_α^n, ρ_n and E_n towards smooth functions, we will need the basic following estimates, valid for ϵ small :

$$\begin{gathered} \|\rho_n(.,t)\|_\infty + \|\nabla_x \rho_n(.,t)\|_\infty + \|\nabla_x^2 \rho_n(.,t)\|_\infty \leq \frac{A}{(1+t)^3}, \quad \forall t > 0 \\ \|\rho_n(.,t)\|_1 + \|\nabla_x \rho_n(.,t)\|_1 \leq A\epsilon \end{gathered} \tag{7.31}$$

where A is a fixed constant which does not depend on n and ϵ. The estimates on ρ_n and $\nabla_x \rho_n$ will allow us to prove that E_n converges strongly in a suitable space where as the estimate $\nabla_x^2 \rho_n$ will give us the regularity of the solution. We shall prove that, if these relations are true at order n, they are also true at order $n+1$, provided the initial data are small enough. To this purpose, we will first consider the case t and use the uniform integrability in v (decay of order $(1+|v|)^{-\alpha}$, $\alpha > 3$) ; then we consider the case t large, and use the uniform integrability in x.

The previous inequality is basically what we need for our theorem 3.2 and as we will notice in the next section, the method to show the bounds for our theorem 3.2 will follow the strategy adopted by Bardos Degond for the previous inequality.

Suppose that we have

$$\Big|\frac{\partial X^n_\alpha}{\partial x}\Big| + \Big|\frac{\partial V^n_\alpha}{\partial x}\Big| + \Big|\frac{\partial^2 X^n_\alpha}{\partial x^2}\Big| + \Big|\frac{\partial^2 V^n_\alpha}{\partial x^2}\Big| + \frac{1}{t}\Big|\frac{\partial V^n_\alpha}{\partial v}\Big| + \frac{1}{t}\Big|\frac{\partial X^n_\alpha}{\partial v}\Big| \leq C \tag{7.32}$$

With

$$u^{n+1}_\alpha(x,v,t) = u_{\alpha,0}(X^n_\alpha(0,t,x,v), V^n_\alpha(0,t,x,v)) \tag{7.33}$$

and the previous relation we also ve have the following bounds :

$$|\rho_{n+1}(x,t)| \leq C\sum_\alpha \int u_{\alpha,0}(X^n_\alpha, V^n_\alpha)dv \tag{7.34}$$

$$|\nabla_x\rho_{n+1}(x,t)| \leq C\sum_\alpha \int |\nabla_{(x,v)}u_{\alpha,0}(X^n_\alpha, V^n_\alpha)|dv \tag{7.35}$$

$$|\nabla^2_x\rho_{n+1}(x,t)| \leq C\sum_\alpha \int |\nabla^2_x u_{\alpha,0}(X^n_\alpha, V^n_\alpha)| + |\nabla_x u_{\alpha,0}(X^n_\alpha, V^n_\alpha)|dv \tag{7.36}$$

So, thanks to the bounds on u_0, we have

$$\int_{\mathbb{R}^3} |\nabla_x\rho_{n+1}(x,t)| \leq \tilde{C}\epsilon \tag{7.37}$$

and eventually replacing A by $max(A,\tilde{C})$ we deduce that the second inequality in is true at order (n+1).

Proposition 7.1. *We assume that E(t,x) satisfies the estimates :*

$$\|\nabla_x E(.,t)\|_\infty \leq \frac{\eta}{(1+t)^{5/2}} \tag{7.38}$$

$$\|\nabla^2_x E(.,t)\|_\infty \leq \frac{\eta}{(1+t)^{5/2}} \tag{7.39}$$

with $\eta < 1$. Then, the following estimates are valid for any (t,x,v) in $\mathbb{R}^+_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v$ and any s such that $0 \leq s \leq t$:

$$\Big|\frac{\partial X}{\partial v}(s,t,x,v) - (s-t)Id\Big| + \Big|\frac{\partial V}{\partial v}(s,t,x,v)\Big| \leq C\eta(t-s) \tag{7.40}$$

$$\Big|\frac{\partial X}{\partial x}(s,t,x,v) - Id\Big| + \Big|\frac{\partial V}{\partial x}(s,t,x,v)\Big| \leq C\eta \tag{7.41}$$

$$\Big|\frac{\partial^2 X}{\partial x^2}(s,t,x,v)\Big| + \Big|\frac{\partial^2 V}{\partial x^2}(s,t,x,v)\Big| \leq C\eta \tag{7.42}$$

where Id denotes the identity matrix of $\mathbb{R}^3$ and C any arbitrary constant.

Proof. For any fixed (t,x,v) in $\mathbb{R}^+_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v$ we write

$$\frac{\partial X}{\partial v}(s) = \frac{\partial X}{\partial v}(s,t,x,v) \quad \text{and} \quad R(s) = \frac{\partial X}{\partial v}(s) - (s-t)Id$$

From the Hamiltonian system we have

$$\frac{\partial \ddot{X}}{\partial v}(s) = \nabla_x E(X(s),s)\cdot\frac{\partial X}{\partial v}(s) \qquad \frac{\partial X}{\partial v}(t) = 0 \qquad \frac{\partial \dot{X}}{\partial v}(t) = Id \tag{7.43}$$

Hence we have $R(t) = 0$ and $\dot{R}(t) = 0$ and thanks to the Taylor formula at the second order, we have

$$R(s) = \int_s^t (u-s)\nabla_x E(X(u),u)\cdot(R(u)+(u-t)Id)du \tag{7.44}$$

and we have

$$|R(s)| \leq \eta \int_s^t \frac{u-s}{(1+u)^{5/2}} |R(u)| du + \eta \int_s^t \frac{(u-s)(t-u)}{(1+u)^{5/2}} du \tag{7.45}$$

An integration by part allow us to show that

$$\int_s^t \frac{(u-s)(t-u)}{(1+u)^{5/2}} du \leq C(t-s) \tag{7.46}$$

and

$$\int_s^u \frac{w-s}{(1+w)^{5/2}} du \leq C \tag{7.47}$$

so by appling the Grönwall inequality and using the fact that $\eta < 1$ we have

$$|R(s)| \leq C\eta(t-s) \tag{7.48}$$

which proves the first inequality. The other inequalities can be shown in a similar way.

□

But for this proposition to hold, we made some assumptions on the Electric field which shall be proven.

Lemma 7.1. *Let $\rho(x)$ be a smooth function belonging to the space $L^1(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$. Then for the function Φ fiven by the formula*

$$\Phi(x) = \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} dy \tag{7.49}$$

one has the following estimates

$$\|\Phi\|_\infty \leq C \|\rho\|_\infty^{1/3} \|\rho\|_1^{2/3} \tag{7.50}$$

$$\|\nabla\Phi\|_\infty \leq C \|\rho\|_\infty^{2/3} \|\rho\|_1^{1/3} \tag{7.51}$$

$$\|D^2\Phi\|_\infty \leq C_\theta \|\rho\|_\infty^{3(1-\theta)/(3+\theta)} \|\nabla_x \rho\|_\infty^{3\theta/(3+\theta)} |\rho\|_1^{\theta/(3+\theta)} \tag{7.52}$$

where C is a constant. D^2 denotes any the second order derivative, and $\theta \in]0,1[$. The last inequality comes from the elliptical regularity of the solution.

Proof. We will just prove the first one and the third one, the second one can be proven in a similar way Let $r > 0$, we have

$$|\Phi(x)| \leq \int_{|x-y|<r} \frac{|\rho(y)|}{|x-y|} dy + \int_{|x-y|>r} \frac{|\rho(y)|}{|x-y|} dy$$

Using the change of variables $\chi = x - y$ and the spherical coordinate for the first term and bound from above by $1/r$ in the second term we have

$$\begin{aligned} |\Phi(x)| &\leq 4\pi \|\rho\|_\infty \int_0^r \frac{\rho^2}{\rho} d\rho + \frac{1}{r} \|\rho\|_1 \\ &\leq 2\pi r^2 \|\rho\|_\infty + \frac{1}{r} \|\rho\|_1 \end{aligned} \tag{7.53}$$

Studying the function on the right hand side of we have

$$|\Phi(x)| \leq \Big(\frac{2\pi}{(4\pi)^{1/3}} + (4\pi)^{1/3} \Big) \|\rho\|_\infty^{2/3} \|\rho\|_1^{1/3}$$

and the inequality follows

For $|D^2\Phi|$ we have for $i \neq j$ ($i = j$ is the same) we have

$$\begin{aligned}
\Big|\frac{\partial^2\Phi}{\partial_{x^i}\partial_{x^j}}\Big| =&\Big|\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^5}\rho(y)dy\Big| \\
=&\Big|\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^5}(\rho(y)-\rho(x))dy\Big| \\
\le& C\Big(\int_{|x-y|<r}\frac{1}{|x-y|^{3-\theta}}\frac{|\rho(y)-\rho(x)|}{|x-y|^\theta}dy+\int_{|x-y|>r}\frac{|\rho(y)|}{|x-y|^3}dy\Big) \\
\le& C_\theta\Big(r^\theta\|\rho\|_{0,\theta}+\frac{1}{r^3}\|\rho\|_1\Big)
\end{aligned} \tag{7.54}$$

where $\|\rho\|_{0,\theta}=\text{Sup}_{x,y}\frac{|\rho(y)-\rho(x)|}{|x-y|^\theta}$
We deduce the relation

$$\Big|\frac{\partial^2\Phi}{\partial_{x_i}\partial_{x_j}}\Big| \le C_\theta\|\rho\|_{0,\theta}^{3/(\theta+3)}\|\rho\|_1^{\theta/(3+\theta)} \tag{7.55}$$

We have for $x,y\in\mathbb{R}^3$:

$$\frac{|\rho(y)-\rho(x)|}{|x-y|^\theta}=\frac{|\rho(y)-\rho(x)|^\theta}{|x-y|^\theta}\cdot|\rho(y)-\rho(x)|^{1-\theta}\le\|\nabla_x\rho\|_\infty^\theta 2\|\rho\|_\infty^{1-\theta}$$

Thus

$$\|\rho\|_{0,\theta}\le 2\|\nabla_x\rho\|_\infty^\theta\|\rho\|_\infty^{1-\theta} \tag{7.56}$$

using the previous inequalities give (7.52) and the proof is complete. □

If we apply the lemma,(with $\theta=3/5$), and use the induction assumption we have :

$$\|E_n(.,t)\|_\infty\le C.A.\frac{\epsilon^{1/3}}{(1+t)^2} \tag{7.57}$$

$$\|\nabla_x E_n(.,t)\|_\infty+\|\nabla_x^2 E_n(.,t)\|_\infty\le C.A.\frac{\epsilon^{1/6}}{(1+t)^{5/2}} \tag{7.58}$$

The proposition 7.1 applies and gives with $\epsilon^{1/6}<1/CA$:

$$\Big|\frac{\partial X_\alpha^n}{\partial x}\Big|+\Big|\frac{\partial V_\alpha^n}{\partial x}\Big|+\Big|\frac{\partial^2 X_\alpha^n}{\partial x^2}\Big|+\Big|\frac{\partial^2 V_\alpha^n}{\partial x^2}\Big|+\frac{1}{t}\Big|\frac{\partial V_\alpha^n}{\partial v}\Big|+\frac{1}{t}\Big|\frac{\partial X_\alpha^n}{\partial v}\Big|\le C \tag{7.59}$$

Now we consider the case $0\le t\le 1$, and we have

$$|V_\alpha^n-v|\le\frac{|q_\alpha|}{m_\alpha}\int_0^t\|E_n(.,s)\|_\infty ds\le CA\epsilon^{1/3} \tag{7.60}$$

Therefore ,

$$|V_\alpha^n|\ge|v|-CA\epsilon^{1/3} \tag{7.61}$$

and from the bounds (7.34) and the bounds on u_0 we obtain

$$|\rho_{n+1}(x,t)|\le C\epsilon\sum_\alpha\int\frac{dv}{(1+|v|-CA\epsilon^{1/3})^4} \tag{7.62}$$

So, for ϵ small

$$\|\rho_{n+1}(.,t)\|_\infty\le\frac{A}{(1+t)^3}\qquad 0\le t\le 1 \tag{7.63}$$

Similarly, the same estimate is valid for $\nabla_x\rho_{n+1}$ and $\nabla_x^2\rho_{n+1}$. We can now consider the case $t\ge 1$.
In order to prove the induction for $t\ge 1$ we have to give a corollary from the proposition we gave above.

Corollary 7.1. Under the assumptions of the previous proposition, there exists a constant η_0 such that, when the vector field $E(x,t)$ satisfies the estimate

$$\|\nabla_x E(.,t)\|_\infty \le \frac{\eta}{(1+t)^{5/2}} \qquad \text{with} \qquad \eta < \eta_0 \tag{7.64}$$

the following facts are true :
1) The determinant of the matrix $\frac{\partial X}{\partial v}(s,t,x,v)$ satisfies the estimate, valid for any (x,v,t) in $\mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$ and any s such that $0 \le s \le t$:

$$\Big| det(\frac{\partial X}{\partial v}(s,t,x,v)) \Big| \ge \frac{(t-s)^3}{2} \tag{7.65}$$

2) For any fixed (s,t,x) $\mathbb{R}_t^+ \times \mathbb{R}_t^+ \times \mathbb{R}_x^3$ such that $0 \le s \le t$, the mapping :

$$\theta : v \to X(s,t,x,v) \tag{7.66}$$

is one to one.

Proof. 1) We have

$$\Big|\det(\frac{\partial X}{\partial v}(s))\Big| = (t-s)^3 \Big|\det\Big(Id + \frac{R(s)}{s-t}\Big)\Big| \tag{7.67}$$

The norm of $R(s)/(s-t)$ goes to zero when η goes to zero uniformly with respect to (s,t,x,v) and by continuity of the mapping determinant 1) follows.

2)We shall keep (t,x) fixed, and we denote by θ_s the mapping for a given s. We first notice that for $s \in [t_*, t]$, with

$$|t - t_*| \le |||\nabla_x E|||^{-1/2} \tag{7.68}$$

the mapping θ_s is one to one. Indeed, suppose that there exists an s_* in $[t_*, t]$ and two distincts values v_1, v_2 such that

$$X(s_*, t, x, v_1) = X(s_*, t, x, v_2) \tag{7.69}$$

Then we will show that the functions $X_i(s) = X(s,t,x,v_i)$ $(i = 1,2)$ which are two solutions of the problem

$$\ddot{X}_i(s) = E(X_i(s), s), \qquad X_i(t) = x \tag{7.70}$$

coincide on the interval $[s_*, t]$. This will prove that $v_1 = v_2$: We denote by $z(s)$ the function

$$z(s) = X_1(s) - X_2(s) \tag{7.71}$$

With an integration by part we then obtain

$$\begin{aligned} \int_{s_*}^t |\dot{z}(s)|^2 ds &\le \int_{s_*}^t |\ddot{z}(s)||z(s)| ds \\ &\le |||\nabla_x E||| \int_{s_*}^t |z(s)|^2 ds \\ &\le |||\nabla_x E|||(t - s_*) \mathrm{Sup}_{s\in[s_*,t]} |z(s)|^2 \end{aligned} \tag{7.72}$$

As we have

$$|z(s)|^2 \le \Big(\int_{s_*}^s \dot{z}(u)du\Big)^2 \le (s - s_*)\Big(\int_{s_*}^s |\dot{z}(u)|^2 du\Big)^2 \tag{7.73}$$

with Cauchy-Schwarz inequality we deduce that

$$\int_{s_*}^t |\dot{z}(s)|^2 ds \le |||\nabla_x E||| \cdot (t - s_*) \int_{s_*}^t |\dot{z}(s)|^2 ds \tag{7.74}$$

This shows that the function $\dot{z}$ is identically zero on the interval $[s_*, t]$. Now, we define a number T by

$$T = \mathrm{Sup}\,\{s \in [0,t] \text{ such that } \theta_s \text{ is not one to one}\} \tag{7.75}$$

From what precedes, we have $T < t$. We will show that such a number does not exist. We denote by $X(s,v)$, the vector $X(s,t,x,v)$ for any fixed (t,x). Let v_1, v_2, $\underline{s}$ such that

$$X(v_1,\underline{s}) = X(v_2,\underline{s}) \tag{7.76}$$

Then thanks to 1), we can apply the implicit function theorem to the function

$$\phi(w,s) = X(w,s) - X(v_1,s)$$

and prove that there exists a function $W(s)$ defined for $|s-\underline{s}| < \epsilon$, such that one has

$$X(v_1,s) = X(W(s),s) \qquad \text{for} \qquad |s-\underline{s}| < \epsilon, \qquad \text{and} \quad W(\underline{s}) = v_2 \tag{7.77}$$

Furthermore, the estimates (7.40) on $\frac{\partial X}{\partial v}$ is independant on (v,s) in $\mathbb{R}^3_v \times [0,t_*]$, ϵ is also independant on v_1, v_2, and $\underline{s}$. Then according to the definition of T, there exists $\underline{s^\epsilon} \in]T-\epsilon/2, T+\epsilon/2[$, and two different values v_1^ϵ and v_2^ϵ such that

$$X(v_1^\epsilon,\underline{s}^\epsilon) = X(v_2^\epsilon,\underline{s}^\epsilon) \tag{7.78}$$

Thanks to what precedes, there exist a function W defined for $|s-\underline{s}^\epsilon| < \epsilon$, verifying (7.77). Now, for s in the interval $]T-\epsilon/2, T+\epsilon/2[$, the points v_1^ϵ and $W(s)$ never coincide otherwise this would violate point 1). Thus T is not an upper bound, which ends the proof of corollary 7.1.

□

Estimates on E and corollary 7.1 proves that for ϵ small, the mapping

$$v \to X_\alpha^n = X_\alpha^n(t,x,v) \tag{7.79}$$

is one to one and that we have

$$\left|det\Big(\frac{\partial X_\alpha^n}{\partial v}\Big)\right| \geq \frac{t^3}{2} \tag{7.80}$$

Therefore we can use the change of variables which appear in the right hand side of

$$|\rho_{n+1}(x,t)| \leq C\sum_\alpha \int u_{\alpha,0}(X_\alpha^n, V_\alpha^n)dv \tag{7.81}$$

$$|\nabla_x\rho_{n+1}(x,t)| \leq C\sum_\alpha \int |\nabla_{(x,v)}u_{\alpha,0}(X_\alpha^n, V_\alpha^n)|dv \tag{7.82}$$

$$|\nabla_x^2\rho_{n+1}(x,t)| \leq C\sum_\alpha \int |\nabla_x^2 u_{\alpha,0}(X_\alpha^n, V_\alpha^n)| + |\nabla_x u_{\alpha,0}(X_\alpha^n, V_\alpha^n)|dv \tag{7.83}$$

With the assumption on u_0 and the estimates given by the corollary this gives

$$\begin{aligned} |\rho_{n+1}(x,t)| &\leq C\epsilon\sum_\alpha \int \frac{dv}{(1+|X_\alpha^n|)^4} \\ &\leq C\epsilon\sum_\alpha \int \frac{1}{(1+|X_\alpha^n|)^4}\left|det\Big(\frac{\partial X_\alpha^n}{\partial v}\Big)\right|^{-1} dX_\alpha^n \\ &\leq C\epsilon/t^3 \\ &\leq A/(1+t^3) \qquad \text{for } \epsilon \text{ small} \end{aligned} \tag{7.84}$$

Similar results for $\nabla_x\rho_{n+1}$ and $\nabla_x^2\rho_{n+1}$ Hence (7.31) is true at order n+1, which completes the iteration.

It is not necessary to continue the proof of Global existence of Vlasov-Poisson to understand our next section on the stress-energy tensor.

8 Estimates for components of the stress energy momentum tensor

The previous section helped us to understand that we will have to use the determinant in order to make a change of variables to have some bounds on the stress-energy tensor T. In this section we shall prove the theorem (3.2). Recall that

$$T^{\mu\nu}(t,x) = \int_{\mathcal{P}_{(t,x)}} f(t,x,p)p^\mu p^\nu \frac{\sqrt{|detg|}}{p^0} dp^1dp^2dp^3$$

8.1 Bounding $Z^I T$ with derivatives of f

Recall that in the previous subsection for the Vlasov-Poisson system we had to use

$$\Big|\frac{\partial X^n_\alpha}{\partial x}\Big| + \Big|\frac{\partial V^n_\alpha}{\partial x}\Big| + \Big|\frac{\partial^2 X^n_\alpha}{\partial x^2}\Big| + \Big|\frac{\partial^2 V^n_\alpha}{\partial x^2}\Big| + \frac{1}{t}\Big|\frac{\partial V^n_\alpha}{\partial v}\Big| + \frac{1}{t}\Big|\frac{\partial X^n_\alpha}{\partial v}\Big| \leq C$$

With

$$u^{n+1}_\alpha(x,v,t) = u_{\alpha,0}(X^n_\alpha(0,t,x,v), V^n_\alpha(0,t,x,v))$$

in order to have the following bounds

$$|\rho_{n+1}(x,t)| \leq C\sum_\alpha \int u_{\alpha,0}(X^n_\alpha, V^n_\alpha)dv$$

$$|\nabla_x \rho_{n+1}(x,t)| \leq C\sum_\alpha \int |\nabla_{(x,v)} u_{\alpha,0}(X^n_\alpha, V^n_\alpha)|dv$$

$$|\nabla^2_x \rho_{n+1}(x,t)| \leq C\sum_\alpha \int |\nabla^2_x u_{\alpha,0}(X^n_\alpha, V^n_\alpha)| + |\nabla_x u_{\alpha,0}(X^n_\alpha, V^n_\alpha)|dv$$

With the section 6 we were able to find some bounds on the vector field applied to the geodesics. We now need to bound $Z^I T^{\mu\nu}$ with derivatives of f. First we will rewrite the integral in the tensor T in terms of $\hat{p}^i$.

Proposition 8.1. *There exists a function Λ, smooth provided $|\hat{p}^i| \leq c < 1$ for $i = 1,2,3$ such that*

$$det\Big(\frac{\partial p^i}{\partial \hat{p}^j}\Big)(t,x,\hat{p}) = \Lambda(\hat{p}, h(t,x)).$$

Proof. We have $\hat{p}^0 = 1$ and since $g_{\alpha\beta}p^\alpha p^\beta = 1$:

$$\sum_{\alpha=0}^3\sum_{\beta=0}^3 g_{\alpha\beta}p^\alpha p^\beta = 1 \qquad \text{Applying } \frac{\partial}{\partial p^j} \qquad \sum_{\alpha=0}^3\sum_{\beta=0}^3 g_{\alpha\beta}\frac{\partial p^\alpha}{\partial p^j}p^\beta + p^\alpha\frac{\partial p^\beta}{\partial p^j} = 0.$$

Hence

$$2\sum_{\alpha=0}^3\sum_{\beta=0}^3 g_{\alpha\beta}p^\alpha\frac{\partial p^\beta}{\partial p^j} = 0 \qquad 2\sum_{\alpha=0}^3\sum_{\beta=1}^3 g_{\alpha\beta}p^\alpha\frac{\partial p^\beta}{\partial p^j} + 2\sum_{\alpha=0}^3 g_{\alpha 0}p^\alpha\frac{\partial p^0}{\partial p^j} = 0$$

Since we have a dummy index in the last sum, in Einstein notations we have

$$\frac{\partial p^0}{\partial p^j} = -\frac{g_{\alpha j}p^\alpha}{g_{\beta 0}p^\beta} = -\frac{g_{\alpha j}\hat{p}^\alpha}{g_{\beta 0}\hat{p}^\beta}$$

Now with $\hat{p}^i = \frac{p^i}{p^0}$ we have

$$\frac{\partial \hat{p}^i}{\partial p^j} = \frac{1}{p^0}(\delta^j_i - \hat{p}^i\frac{\partial p^0}{\partial p^j}) = \frac{1}{p^0}(\delta^j_i + \hat{p}^i\frac{g_{\alpha j}\hat{p}^\alpha}{g_{\beta 0}\hat{p}^\beta})$$

Since $g_{\alpha\beta}p^\alpha p^\beta = 1$ we have

$$p^0 = \sqrt{-\frac{1}{g_{\alpha\beta}\hat{p}^\alpha\hat{p}^\beta}}$$

Writing $g_{\alpha\beta} = m_{\alpha\beta} + h_{\alpha\beta}$ and $det(A^{-1}) = det(A)^{-1}$ the proposition follows. ☐

Proposition 8.2. *There exists smooth functions $\Lambda^{\mu\nu}$, smooth provided $|\hat{p}| \leq c < 1$ such that*

$$T^{\mu\nu}(t,x) = \int_{\mathcal{P}_{(t,x)}} f(t,x,\hat{p})\Lambda^{\mu\nu}(\hat{p}, h(t,x))d\hat{p}^1 d\hat{p}^2 d\hat{p}^3$$

Proof. In Minkowski's space, we have $h = 0$ and if we define p^0_M by the relation $m_{\alpha\beta}p^\alpha p^\beta = -1$,writing $\hat{p}^j_M = \frac{p^j}{p^0_M}$ after some computation we have

$$\frac{\partial p^i}{\partial \hat{p}^j_M} = (p^0_M)^5 = (1 + (p^1)^2 + (p^2)^2 + (p^3)^2)^{5/2}$$

Since $|\hat{p}^i| \leq c < 1$

$$\Big|\frac{\partial p^i}{\partial \hat{p}^j} - (1+(p^1)^2+(p^2)^2+(p^3)^2)^{5/2}\Big| = \Big|\Lambda(\hat{p},h(t,x)) - \Lambda(\hat{p},0)\Big| \leq \sup|\partial\Lambda||h(t,x)| \leq C\epsilon$$

Hence, for ϵ small enough the change of variables $(t,x,p) \to (t,x,\hat{p})$ is well defined. Using again

$$p^0 = \sqrt{-\frac{1}{g_{\alpha\beta}\hat{p}^\alpha\hat{p}^\beta}}$$

we have

$$p^\mu p^\nu \frac{\sqrt{|detg|}}{p^0} det\Big(\frac{\partial p^i}{\partial \hat{p}^j}\Big)(t,x,\hat{p}) = \Lambda^{\mu\nu}(\hat{p},h(t,x))$$

for some smooth functions $\Lambda^{\mu\nu}$, smooth provided $|\hat{p}| \leq c < 1$. The proposition follows

□

In subsection 5.4 we defined the modified vector fields and gave their expression with some $\mathring{Z}^k(t,x,\hat{p})$ which have the form

$$\mathring{Z}^k(t,x,\hat{p}) = \mathring{Z}^k_{1,l}(t,x)\hat{p}^l + \mathring{Z}^k_2(t,x).$$

Recall that

$$\mathring{\Omega}^k_{ij}|_{(t,x,p)} = \Theta^i(t,x)\delta^k_j - \Theta^j(t,x)\delta^k_i - \Omega_{ij}(\Theta^k(t,x))$$

$$\mathring{B}^k_i|_{(t,x,p)} = -\Theta^i(t,x)\hat{p}^k - B_i(\Theta^k(t,x)) - \frac{x^i}{t-t_0}\Theta^k(t,x)$$

$$\mathring{S}^k|_{(t,x,p)} = \Theta^k(t,x) - S(\Theta^k(t,x)) - \frac{x^t}{t-t_0}\Theta^k(t,x)$$

Hence

$$\mathring{\Omega}^k_{ij,1,l}(t,x) = 0 \qquad \mathring{\Omega}^k_{ij,2}(t,x) = \Theta^i(t,x)\delta^k_j - \Theta^j(t,x)\delta^k_i - \Omega_{ij}(\Theta^k(t,x))$$

$$\mathring{B}^k_{i,1,l}(t,x) = -\Theta^i(t,x)\delta^k_l \qquad \mathring{B}^k_{i,2}(t,x) = -B_i(\Theta^k(t,x)) - \frac{x^i}{t-t_0}\Theta^k(t,x)$$

$$\mathring{S}^k_{1,l}(t,x) = 0 \qquad \mathring{S}^k_2(t,x) = \Theta^k(t,x) - S(\Theta^k(t,x)) - \frac{x^t}{t-t_0}\Theta^k(t,x)\Theta^k(t,x)$$

Proposition 8.3. *For $\mu,\nu = 0,1,2,3$ and any multi index I, there exist functions Λ such that,*

$$\begin{aligned} Z^I T^{\mu\nu}(t,x) = &\sum_{|I_1|+|I_2|\leq|I|} \int \overline{Z}^{I_1}(f(t,x,\hat{p})) \times \sum_{k=0}^{|I_2|} \sum_{|J_1|+\ldots+|J_k|\leq|I_2|,\ |J_i|\geq 1} \\ &\sum_{|L_1|+\ldots+|L_k|\leq|I_2|-1,\ |L_i|\geq1} \Big(\overline{Z}^{J_1}(\hat{p}^{l_1}) + \overline{Z}^{J_1}(h_{\alpha_1\beta_1})\Lambda^{l_1,\alpha_1,\beta_1}(\hat{p},h) + \overline{Z}^{L_1}(\mathring{Z}^m_{1,m})\Lambda^{l_1}(\hat{p},h)\Big)\times \\ &\cdots \times \Big(\overline{Z}^{J_k}(\hat{p}^{l_k}) + \overline{Z}^{J_k}(h_{\alpha_k\beta_k})\Lambda^{l_k,\alpha_k,\beta_k}(\hat{p},h) + \overline{Z}^{L_k}(\mathring{Z}^m_{1,m})\Lambda^{l_k}(\hat{p},h)\Big)\Lambda(\hat{p},h)d\hat{p} \end{aligned}$$

Proof. We shall prove the result by induction. For $|I| = 1$, we have

$$\overline{Z}(\Lambda^{\mu\nu}(\hat{p},h(t,x))) = \overline{Z}(\hat{p}^l)(\partial_{\hat{p}^l}\Lambda^{\mu\nu})(\hat{p},h(t,x)) + \sum_{\alpha\beta}(Zh_{\alpha\beta})(t,x)(\partial_{h_{\alpha\beta}}\Lambda^{\mu\nu})(\hat{p},h(t,x)),$$

Moreover, for any function $\eta(t,x,\hat{p})$, an integration by part allow us to show that

$$Z\Big(\int \eta(t,x,\hat{p})\Big) = \int (\overline{Z} - \mathring{Z}^k\partial_{\hat{p}^k})\eta(t,x,\hat{p})d\hat{p} = \int \overline{Z}(\eta(t,x,\hat{p})) + (\partial_{\hat{p}^k}\mathring{Z}^k)(t,x)\eta(t,x,\hat{p})d\hat{p}$$

We also have

$$(\partial_{\hat{p}^k}\mathring{Z}^k)(t,x) = \mathring{Z}^k_{1,k}(t,x)$$

So for $|I| = 1$ and $Z = \Omega_{ij}, B_i, S$ we have

$$ZT^{\mu\nu}(t,x) = \int \big(\overline{Z}(f(t,x,\hat{p})) + \mathring{Z}^m_{1,m}(t,x) f(t,x,\hat{p})\big)\Lambda^{\mu\nu}(\hat{p},h(t,x)) +$$
$$f(t,x,\hat{p})\big(\overline{Z}(\hat{p}^l)(\partial_{\hat{p}^l}\Lambda^{\mu\nu})(\hat{p},h(t,x)) + (Zh_{\alpha\beta})(t,x)(\partial_{h_{\alpha\beta}}\Lambda^{\mu\nu})(\hat{p},h(t,x))\big)d\hat{p}$$

Hence

$$ZT^{\mu\nu}(t,x) = \int \overline{Z}(f(t,x,\hat{p}))\Lambda^{\mu\nu}(\hat{p},h(t,x)) +$$
$$f(t,x,\hat{p})\big(\overline{Z}(\hat{p}^l)(\partial_{\hat{p}^l}\Lambda^{\mu\nu})(\hat{p},h(t,x)) + (Zh_{\alpha\beta})(t,x)(\partial_{h_{\alpha\beta}}\Lambda^{\mu\nu})(\hat{p},h(t,x)) + \mathring{Z}^m_{1,m}(t,x)\Lambda^{\mu\nu}(\hat{p},h(t,x))\big)d\hat{p}$$

The proof for $|I| = 1$ follows.
Suppose now that the relation holds at $|I|$. Applying Z to $Z^I T^{\mu\nu}$ we have $Z^{I+1}T^{\mu\nu}(t,x) = \sum_{|I_1+|I_2|\leq|I|} Z \int A(t,x,\hat{p})d\hat{p}$ where

$$A(t,x,\hat{p}) = \overline{Z}^{I_1}(f(t,x,\hat{p})) \times \sum_{k=0}^{|I_2|} \sum_{|J_1|+\ldots+|J_k|\leq|I_2|,\ |J_i|\geq1}$$
$$\sum_{|L_1|+\ldots+|L_k|\leq|I_2|-1,\ |L_i|\geq1} \Big(\overline{Z}^{J_1}(\hat{p}^{l_1}) + \overline{Z}^{J_1}(h_{\alpha_1\beta_1})\Lambda^{l_1,\alpha_1,\beta_1}(\hat{p},h) + \overline{Z}^{L_1}(\mathring{Z}^m_{1,m})\Lambda^{l_1}(\hat{p},h)\Big)\times$$
$$\cdots\times\Big(\overline{Z}^{J_k}(\hat{p}^{l_k}) + \overline{Z}^{J_k}(h_{\alpha_k\beta_k})\Lambda^{l_k,\alpha_k,\beta_k}(\hat{p},h) + \overline{Z}^{L_k}(\mathring{Z}^m_{1,m})\Lambda^{l_k}(\hat{p},h)\Big)\Lambda(\hat{p},h)d\hat{p}$$

Hence

$$Z^{I+1}T^{\mu\nu}(t,x) = \sum_{|I_1+|I_2|\leq|I|} \int \overline{Z}(A(t,x,\hat{p})) + \mathring{Z}^k_{1,k}(t,x)A(t,x,\hat{p})d\hat{p}$$

For clarity purposes we will often write just $\sum$ instead of the whole sum until the end of the proof. Now we have

$$\overline{Z}(A(t,x,\hat{p})) = \overline{Z}^{R_1}(f(t,x,\hat{p}))\sum\sum\sum$$
$$+\overline{Z}^{I_1}(f(t,x,\hat{p}))\sum\sum\sum\overline{Z}\Big(\prod_{u=1}^{k}\underbrace{\Big(\overline{Z}^{J_u}(\hat{p}^{l_u}) + \overline{Z}^{J_u}(h_{\alpha_u\beta_u})\Lambda^{l_u,\alpha_u,\beta_u}(\hat{p},h) + \overline{Z}^{L_u}(\mathring{Z}^m_{1,m})\Lambda^{l_u}(\hat{p},h)\Big)}_{g_u}\Lambda(\hat{p},h)\Big)$$

where $|R_1| = |I_1| + 1$ and

$$\overline{Z}\Big(\prod_{u=1}^{k} g_u\Big) = \sum_{u=1}^{k}\overline{Z}(g_u)\prod_{s\neq u} g_s$$

Moreover

$$\overline{Z}(g_u) = \overline{Z}^{K_u}(\hat{p}^{l_u}) + \overline{Z}^{K_u}(h_{\alpha_u\beta_u})\Lambda^{l_u,\alpha_u,\beta_u}(\hat{p},h) + \overline{Z}^{J_u}(h_{\alpha_u\beta_u})\overline{Z}(\Lambda^{l_u,\alpha_u,\beta_u}(\hat{p},h)) +$$
$$\overline{Z}^{T_u}(\mathring{Z}^m_{1,m})\Lambda^{l_u}(\hat{p},h) + \overline{Z}^{L_u}(\mathring{Z}^m_{1,m})\overline{Z}(\Lambda^{l_u}(\hat{p},h))$$

where $|K_u| = |J_u| + 1$ and $|T_u| = |L_u| + 1$
Hence we have

$$Z^{I+1}T^{\mu\nu}(t,x) = \sum_{|I_1+|I_2|\leq|I|}\int \overline{Z}^{R_1}(f(t,x,\hat{p}))\sum\sum\sum$$
$$+\overline{Z}^{I_1}(f(t,x,\hat{p}))\Big[\sum\sum\sum\sum_{u=1}^{k}\big(\overline{Z}^{K_u}(\hat{p}^{l_u}) + \overline{Z}^{K_u}(h_{\alpha_u\beta_u})\Lambda^{l_u,\alpha_u,\beta_u}(\hat{p},h) + \overline{Z}^{J_u}(h_{\alpha_u\beta_u})\overline{Z}(\Lambda^{l_u,\alpha_u,\beta_u}(\hat{p},h)) +$$
$$\overline{Z}^{T_u}(\mathring{Z}^m_{1,m})\Lambda^{l_u}(\hat{p},h) + \overline{Z}^{L_u}(\mathring{Z}^m_{1,m})\overline{Z}(\Lambda^{l_u}(\hat{p},h))\big)\prod_{s\neq u}(g_s)\Lambda(\hat{p},h)$$
$$+\prod_{u=1}^{k}(g_u)\overline{Z}(\Lambda(\hat{p},h)) + \mathring{Z}^k_{1,k}(t,x)\sum\sum\sum\Big]d\hat{p}$$

Now expanding every factor gives the induction at order $|I|+1$

□

Proposition 8.4. *For any multi index I, there exists constants $C_{I,k,J,L}$ such that,*

$$\overline{Z}^I(f(t,x,\hat{p})) = \sum_{k+m=1}^{|I|} \sum_{\substack{|J_1|+...+|J_k|+|L_1|+...+|L_m|\leq|I|,\\ |J_i|\geq 1,\ |L_i|\geq 1}} \overline{Z}^{J_1}\left(X(t_0)^{i_1}\right)\cdots\overline{Z}^{J_k}\left(X(t_0)^{i_k}\right)$$
$$\overline{Z}^{L_1}\left(\hat{P}(t_0)^{l_1}\right)\overline{Z}^{L_m}\left(\hat{P}(t_0)^{l_m}\right)\times C_{I,k,J,L}(\partial_{x^{i_1}}...\partial_{x^{i_1}}\partial_{\hat{p}^{l_1}}...\partial_{\hat{p}^{l_m}}f)(t_0,X(t_0),\hat{P}(t_0))$$

Proof. According to the Vlasov equation we have

$$\overline{Z}f(t,x,\hat{p}) = \overline{Z}(X(t_0,t,x,\hat{p})^i)(\partial_{x_i}f)(t_0,X(t_0),\hat{P}(t_0)) + \overline{Z}(\hat{P}(t_0,t,x,\hat{p})^i)(\partial_{\hat{p}_i}f)(t_0,X(t_0),\hat{P}(t_0)) \tag{8.1}$$

The case $|I|=1$ is then proved. The proof follows from a straightforward induction □

Thanks to this proposition and the previous one and the paper from Bardos-Degond it is clear to see what results we want to have. We need to have some bounds of the integral with derivatives of f applied to our Hamiltonian system and use a change of variables. In order to do that we shall prove the following proposition :

Proposition 8.5. *Suppose $t\geq t_0+1$, $|x|\leq ct$, and the bounds (4.1) hold. Then, for each $\mu,\nu=0,1,2,3$ and any multi index I with $|I|\leq N$ and Z^I equal to a product of $|I|$ of the vector fields Ω_{ij}, B_i, S,*

$$|Z^I T^{\mu\nu}(t,x)| \leq C\sum_{n_1+n_2\leq|I|}\int|(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p} + C\sum_{n_1+n_2\leq\lfloor\frac{|I|}{2}\rfloor+1}\int|(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))|$$
$$\times\Big(t^{2-a}\sum_{|J|\leq|I|-1}|(Z^J\Gamma)(t,x)| + \sum_{|J|\leq|I|}\int_{t_0}^t(s)^{1-a}\Big|(Z^J\Gamma)(s,s\frac{x}{t})\Big| + s\Big|(Z^J\Gamma)(s,s\frac{x}{t}) - (Z^J\Gamma)(s,X(s))\Big|ds\Big)d\hat{p}$$

Proof. From proposition 8.3 we have

$$Z^I T^{\mu\nu}(t,x) = \sum_{|I_1+|I_2|\leq|I|}\int\overline{Z}^{I_1}(f(t,x,\hat{p}))\times\sum_{k=0}^{|I_2|}\sum_{|J_1|+...+|J_k|\leq|I_2|,\ |J_i|\geq 1}$$
$$\sum_{|L_1|+...+|L_k|\leq|I_2|-1,\ |L_i|\geq 1}\Big(\overline{Z}^{J_1}(\hat{p}^{l_1}) + \overline{Z}^{J_1}(h_{\alpha_1\beta_1})\Lambda^{l_1,\alpha_1,\beta_1}(\hat{p},h) + \overline{Z}^{L_1}(\mathring{Z}^m_{1,m})\Lambda^{l_1}(\hat{p},h)\Big)\times$$
$$\cdots\times\Big(\overline{Z}^{J_k}(\hat{p}^{l_k}) + \overline{Z}^{J_k}(h_{\alpha_k\beta_k})\Lambda^{l_k,\alpha_k,\beta_k}(\hat{p},h) + \overline{Z}^{L_k}(\mathring{Z}^m_{1,m})\Lambda^{l_k}(\hat{p},h)\Big)\Lambda(\hat{p},h)d\hat{p}$$

For $|I_1|\geq\lfloor\frac{|I|}{2}\rfloor+1$ where I_1 is the multi index that appear in the previous sum, we have $|I_2|\leq\lfloor\frac{|I|}{2}\rfloor+1\leq\lfloor\frac{N}{2}\rfloor+1$. Then for every J_i we have

$$|\overline{Z}^{J_i}(h_{\alpha_i\beta_i})(t,x)|\leq C$$

Proposition 6.5 gives

$$|\overline{Z}^{J_i}(\hat{p}^i)|\leq C$$

Moreover, proposition 6.4 implies that for every $1\leq|L_i|\leq|I_2|-1$,

$$|\overline{Z}^{L_i}(\mathring{Z}^k_{1,k})|\leq C$$

for each $\mathring{Z}^k_{1,k}$.

Now from proposition 8.4 we have

$$\overline{Z}^I(f(t,x,\hat{p})) = \sum_{k+m=1}^{|I|} \sum_{\substack{|J_1|+...+|J_k|+|L_1|+...+|L_m|\leq|I|,\\ |J_i|\geq 1,\ |L_i|\geq 1}} \overline{Z}^{J_1}\left(X(t_0)^{i_1}\right)\cdots\overline{Z}^{J_k}\left(X(t_0)^{i_k}\right)$$
$$\overline{Z}^{L_1}\left(\hat{P}(t_0)^{l_1}\right)\cdots\overline{Z}^{L_m}\left(\hat{P}(t_0)^{l_m}\right)\times C_{I,k,J,L}(\partial_{x^{i_1}}...\partial_{x^{i_1}}\partial_{\hat{p}^{l_1}}...\partial_{\hat{p}^{l_m}}f)(t_0,X(t_0),\hat{P}(t_0))$$

If $1 \leq k+m \leq |I_1|$, $|J_1|+...+|J_k|+|L_1|+...+|L_k| \leq |I_1|$, $|J_i| \geq 1$, $|L_i| \geq 1$. If $k+m \geq \lfloor \frac{|I_1|}{2} \rfloor + 2$ then $|J_i| \leq \lfloor \frac{|I_1|}{2} \rfloor + 1 \leq \lfloor \frac{|N|}{2} \rfloor + 1$ for $i = 1, ..., k$ and $|J_i| \leq \lfloor \frac{|L_1|}{2} \rfloor + 1 \leq \lfloor \frac{|N|}{2} \rfloor + 1$ for $i = 1, ..., m$.

From proposition 6.11 we obtain

$$|\overline{Z}^{J_1}\big(X(t_0)^{i_1}\big) \cdots \overline{Z}^{J_k}\big(X(t_0)^{i_k}\big)\overline{Z}^{L_1}\big(\hat{P}(t_0)^{l_1}\big) \cdots \overline{Z}^{L_m}\big(\hat{P}(t_0)^{l_m}\big)$$
$$\times C_{I,k,J,L}(\partial_{x^{i_1}}...\partial_{x^{i_1}}\partial_{\hat{p}^{l_1}}...\partial_{\hat{p}^{l_m}} f)(t_0, X(t_0), \hat{P}(t_0))| \leq C \sum_{n_1+n_2 \leq |I_1|} |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0, X(t_0), \hat{P}(t_0))|$$

Now from Corollary 6.4 we have for $k+m \leq \lfloor \frac{|I_1|}{2} \rfloor + 1$ there is at most one i such that $|J_i| \geq \lfloor \frac{|I_1|}{2} \rfloor + 2$ or $|L_i| \geq \lfloor \frac{|I_1|}{2} \rfloor + 2$ and thus :

$$|\overline{Z}^{J_1}\big(X(t_0)^{i_1}\big) \cdots \overline{Z}^{J_k}\big(X(t_0)^{i_k}\big)\overline{Z}^{L_1}\big(\hat{P}(t_0)^{l_1}\big) \cdots \overline{Z}^{L_m}\big(\hat{P}(t_0)^{l_m}\big)$$
$$\times C_{I,k,J,L}(\partial_{x^{i_1}}...\partial_{x^{i_1}}\partial_{\hat{p}^{l_1}}...\partial_{\hat{p}^{l_m}} f)(t_0, X(t_0), \hat{P}(t_0))|$$
$$\leq C \sum_{n_1+n_2 \leq |I_1|} |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0, X(t_0), \hat{P}(t_0))|\Big(1 + t^{2-a} \sum_{|J| \leq |I|-1} |(Z^J\Gamma)(t,x)|$$
$$+ \sum_{|J| \leq |I|} \int_{t_0}^t (s')^{1-a}\Big|(Z^J\Gamma)(s', s'\frac{x}{t})\Big| + s'\Big|(Z^J\Gamma)(s', s'\frac{x}{t}) - (Z^J\Gamma)(s', X(s'))\Big|ds'\Big)$$

Hence

$$\int \overline{Z}^{I_1}(f(t,x,\hat{p})) \times \sum_{k=0}^{|I_2|} \sum_{|J_1|+...+|J_k| \leq |I_2|,\, |J_i| \geq 1}$$
$$\sum_{|L_1|+...+|L_k| \leq |I_2|-1,\, |L_i| \geq 1} \Big(\overline{Z}^{J_1}(\hat{p}^{l_1}) + \overline{Z}^{J_1}(h_{\alpha_1\beta_1})\Lambda^{l_1,\alpha_1,\beta_1}(\hat{p},h) + \overline{Z}^{L_1}(\mathring{Z}^m_{1,m})\Lambda^{l_1}(\hat{p},h)\Big) \times$$
$$\cdots \times \Big(\overline{Z}^{J_k}(\hat{p}^{l_k}) + \overline{Z}^{J_k}(h_{\alpha_k\beta_k})\Lambda^{l_k,\alpha_k,\beta_k}(\hat{p},h) + \overline{Z}^{L_k}(\mathring{Z}^m_{1,m})\Lambda^{l_k}(\hat{p},h)\Big)\Lambda(\hat{p},h)d\hat{p}$$
$$\leq C \sum_{n_1+n_2 \leq |I|} \int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0, X(t_0), \hat{P}(t_0))|d\hat{p} + C \sum_{n_1+n_2 \leq \lfloor \frac{|I|}{2} \rfloor + 1} \int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0, X(t_0), \hat{P}(t_0))| \times$$
$$\Big(t^{2-a} \sum_{|J| \leq |I|-1} |(Z^J\Gamma)(t,x)| + \sum_{|J| \leq |I|} \int_{t_0}^t s^{1-a}\Big|(Z^J\Gamma)(s, s\frac{x}{t})\Big| + s\Big|(Z^J\Gamma)(s, s\frac{x}{t}) - (Z^J\Gamma)(s, X(s))\Big|ds\Big)d\hat{p}$$

Now if we suppose that $|I_2| \geq \lfloor \frac{|I|}{2} \rfloor + 1$, use the propositions 8.3, 8.4 along with the propositions 6.4 and 6.5 one can show in a similar way that

$$\int \overline{Z}^{I_1}(f(t,x,\hat{p})) \times \sum_{k=0}^{|I_2|} \sum_{|J_1|+...+|J_k| \leq |I_2|,\, |J_i| \geq 1}$$
$$\sum_{|L_1|+...+|L_k| \leq |I_2|-1,\, |L_i| \geq 1} \Big(\overline{Z}^{J_1}(\hat{p}^{l_1}) + \overline{Z}^{J_1}(h_{\alpha_1\beta_1})\Lambda^{l_1,\alpha_1,\beta_1}(\hat{p},h) + \overline{Z}^{L_1}(\mathring{Z}^m_{1,m})\Lambda^{l_1}(\hat{p},h)\Big) \times$$
$$\cdots \times \Big(\overline{Z}^{J_k}(\hat{p}^{l_k}) + \overline{Z}^{J_k}(h_{\alpha_k\beta_k})\Lambda^{l_k,\alpha_k,\beta_k}(\hat{p},h) + \overline{Z}^{L_k}(\mathring{Z}^m_{1,m})\Lambda^{l_k}(\hat{p},h)\Big)\Lambda(\hat{p},h)d\hat{p}$$
$$\leq C \sum_{n_1+n_2 \leq |I|} \int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0, X(t_0), \hat{P}(t_0))|d\hat{p}\Big(1 + \sum_{|J| \leq |I_2|-1} t|(Z^J\Gamma)(t,x)| + \sum_{|J| \leq |I_2|} \int_{t_0}^t |(Z^J\Gamma)(s', s'\frac{x}{t})|ds'\Big)d\hat{p}$$

The propositon follows.

□

8.2 Determinants and changes of variables

Now that we estimated the components of our vector fields applied to the stress energy tensor in terms of derivative of f applied to the geodesics thanks to the proposition 8.5 we can now use the same ideas as in Bardos Degond paper to prove the theorem 3.2. Hence we will use for instance the change of variables : $(x,t,\hat{p}) \to (t,x,y)$ where

$$y^i(t,x,\hat{p}) = X(t_0,t,x,\hat{p})^i$$

for i=1,2,3.

Lemma 8.1. *Suppose $t \geq t_0$, $|x| \leq ct$, and $(t,x,\hat{p}) \in supp(f)$. Then, if the assumption (6.1) holds and ϵ is sufficiently small,*

$$\Big|\frac{\partial X^i}{\partial \hat{p}^j}(s,t,x,\hat{p}) + (t-s)\delta^i_j\Big| \lesssim \frac{\epsilon t}{s^a}, \qquad \Big|\frac{\partial \hat{P}^i}{\partial \hat{p}^j}(s,t,x,\hat{p}) - \delta^i_j\Big| \lesssim \frac{\epsilon t}{s^{1+a}} \tag{8.2}$$

and

$$\Big|\frac{\partial X^i}{\partial x^j}(s,t,x,\hat{p}) - \delta^i_j\Big| \lesssim \frac{\epsilon}{s^a}, \qquad \Big|\frac{\partial \hat{P}^i}{\partial x^j}(s,t,x,\hat{p})\Big| \lesssim \frac{\epsilon}{s^{1+a}} \tag{8.3}$$

for all $t_0 \leq s \leq t$ and $i,j = 1,2,3$

This lemma should be compared with the proposition 7.1 in the paper from Claude Bardos and Pierre Degond [2].

Proof. If we try the same idea as in Bardos Degond and we call

$$\phi(s) = \frac{\partial X^i}{\partial \hat{p}^j}(s,t,x,\hat{p}) + (t-s)\delta^i_j$$

we get $\phi(t) = 0$, $\dot{\phi}(t) = 0$ and

$$\ddot{\phi(s)} = \nabla_x \hat{\Gamma}^i(s, X^k(s), \hat{P}(s))\frac{\partial X^k}{\partial \hat{p}^j}(s) + \nabla_p \hat{\Gamma}^i(s, X(s), \hat{P}^k(s))\frac{\partial \hat{P}^k}{\partial \hat{p}^j}(s)$$

With the estimates :

$$s^{1+a}|\Gamma^\alpha_{\beta\gamma}(s,X(s))| + s^{2+a}|\partial_k \Gamma^\alpha_{\beta\gamma}(s,X(s))| \lesssim \epsilon$$

we have

$$|\ddot{\phi(s)}| \lesssim \epsilon \sum_{k,l=1}^3 \Big(\frac{1}{s^{2+a}}\Big|\frac{\partial X^k}{\partial \hat{p}^l}(s)\Big| + \frac{1}{s^{1+a}}\Big|\frac{\partial \hat{P}^k}{\partial \hat{p}^l}(s)\Big|\Big)$$

If we integrate this inequality between s and t and since $\frac{\partial \hat{P}^k}{\partial \hat{p}^l}(t,t,x,\hat{p}) = \delta^i_j$ we have :

$$\begin{aligned}\Big|\frac{\partial \hat{P}^i}{\partial \hat{p}^j}(s,t,x,\hat{p}) - \delta^i_j\Big| &\lesssim \epsilon \sum_{k,l=1}^3 \int_s^t \Big(\frac{1}{\tilde{s}^{2+a}}\Big|\frac{\partial X^k}{\partial \hat{p}^l}(\tilde{s})\Big| + \frac{1}{\tilde{s}^{1+a}}\Big|\frac{\partial \hat{P}^k}{\partial \hat{p}^l}(\tilde{s})\Big| d\tilde{s} \\ &\lesssim \frac{\epsilon}{s^a} + \epsilon \sum_{k,l=1}^3 \int_s^t \Big(\frac{1}{\tilde{s}^{2+a}}\Big|\frac{\partial X^k}{\partial \hat{p}^l}(\tilde{s})\Big| + \frac{1}{\tilde{s}^{1+a}}\Big|\frac{\partial \hat{P}^k}{\partial \hat{p}^l}(\tilde{s}) - \delta^k_l\Big| d\tilde{s}\end{aligned}$$

Now if we divide by s^{1+a}, integrate between s and t and sum over i, j we have :

$$\begin{aligned}\sum_{k,l=1}^3 \int_s^t \frac{1}{(s')^{1+a}}\Big|\frac{\partial \hat{P}^k}{\partial \hat{p}^l}(s') - \delta^k_l\Big| ds' &\lesssim \int_s^t \frac{1}{(s')^{1+a}}\Big(\frac{\epsilon}{(s')^a} + \epsilon \sum_{k,l=1}^3 \int_{s'}^t \Big(\frac{1}{\tilde{s}^{2+a}}\Big|\frac{\partial X^k}{\partial \hat{p}^l}(\tilde{s})\Big| + \frac{1}{\tilde{s}^{1+a}}\Big|\frac{\partial \hat{P}^k}{\partial \hat{p}^l}(\tilde{s}) - \delta^k_l\Big| d\tilde{s}\Big) ds' \\ &\lesssim \frac{\epsilon}{s^a} + \epsilon \sum_{k,l=1}^3 \int_s^t \Big(\frac{1}{\tilde{s}^{2+a}}\Big|\frac{\partial X^k}{\partial \hat{p}^l}(\tilde{s})\Big| + \frac{1}{\tilde{s}^{1+a}}\Big|\frac{\partial \hat{P}^k}{\partial \hat{p}^l}(\tilde{s}) - \delta^k_l\Big| d\tilde{s}\end{aligned}$$

Hence with ϵ small enough we get

$$\sum_{k,l=1}^3 \int_s^t \frac{1}{(s')^{1+a}}\Big|\frac{\partial \hat{P}^k}{\partial \hat{p}^l}(s') - \delta^k_l\Big| ds' \lesssim \frac{\epsilon}{s^a} + \epsilon \sum_{k,l=1}^3 \int_s^t \frac{1}{\tilde{s}^{2+a}}\Big|\frac{\partial X^k}{\partial \hat{p}^l}(\tilde{s})\Big| d\tilde{s}$$

We now have

$$\Big|\frac{\partial \hat{P}^i}{\partial \hat{p}^j}(s,t,x,\hat{p}) - \delta^i_j\Big| \lesssim \frac{\epsilon}{s^a} + \epsilon \sum_{k,l=1}^3 \int_s^t \frac{1}{\tilde{s}^{2+a}}\Big|\frac{\partial X^k}{\partial \hat{p}^l}(\tilde{s})\Big| d\tilde{s} \tag{8.4}$$

Integrating betwen s and t :

$$|\phi(s)| \lesssim \frac{\epsilon t}{s^a} + \epsilon \sum_{k,l=1}^{3} \int_s^t \frac{1}{\tilde{s}^{1+a}} \Big| \frac{\partial X^k}{\partial \hat{p}^l}(\tilde{s}) + (t-s)\delta_l^k \Big| d\tilde{s}$$

Dividing by s^{1+a}, integrating between s and t, summing over i, j, taking ϵ small enough as we already did above give :

$$|\phi(s)| \lesssim \frac{\epsilon t}{s^a}$$

Inserting this bound in (8.4) give the result (8.2). The other bounds follow in a similar way.

□

Proposition 8.6. *For fixed t, x with $t \geq t_0+1$, $|x| \leq ct$, if the assumptions (6.1) hold and ϵ is sufficiently small, then, for $\hat{p}$ such that $(t,x,\hat{p}) \in supp(f)$, the change of variables $\hat{p} \mapsto y := X(t_0,t,x,\hat{p})$ satisfies*

$$\Big| det\Big(\frac{\partial \hat{p}^i}{\partial y^j}\Big)^{-1} \Big| \leq \frac{C}{t^3} \tag{8.5}$$

Define

$$z_1^i(s,t,x,y) := X(s,t,x,\hat{p}(t,x,y))^i, \qquad z_2^i(\sigma,s,t,x,y) := \sigma s \frac{x^i}{t} + (1-\sigma) X(s,t,x,\hat{p}(t,x,y))^i,$$

for i=1,2,3. If $t \geq t_0+1$ and $t_0 \leq s \leq t_0 + \frac{1}{2}$, then the change of variables $(x,y) \mapsto (x, z_1(s,t,x,y))$ satisfies,

$$\Big| det\Big(\frac{\partial z_1^i(s,t)}{\partial y^j}\Big)^{-1} \Big| \leq C \tag{8.6}$$

If $t_0 + \frac{1}{2} \leq s \leq t$ then the change of variables $(x,y) \mapsto (z_1(s,t,x,y), y)$ satisfies

$$\Big| det\Big(\frac{\partial z_1^i(s,t)}{\partial x^j}\Big)^{-1} \Big| \leq C\Big(\frac{t}{s}\Big)^3 \tag{8.7}$$

Finally if $t_0 + \frac{1}{2} \leq s \leq t$ and $0 \leq \sigma \leq 1$, then the change of variables $(x,y) \mapsto (z_2(\sigma,s,t,x,y), y)$ satisfies

$$\Big| det\Big(\frac{\partial z_2^i(\sigma,s,t)}{\partial x^j}\Big)^{-1} \Big| \leq C\Big(\frac{t}{s}\Big)^3 \tag{8.8}$$

Moreover, for $t \geq t_0$, the determinant of the 6 by 6 matrix κ satisfies,

$$|det\, \kappa(t_0,t,x,\hat{p}) - 1| \lesssim \epsilon, \qquad \text{where } \kappa = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial \hat{p}} \\ \frac{\partial \hat{P}}{\partial x} & \frac{\partial \hat{P}}{\partial \hat{p}} \end{pmatrix} \tag{8.9}$$

Proof. Recall the bound (8.2) from the previous proposition. Setting $s = t_0$ in this bound we have

$$\Big| \frac{\partial y^i}{\partial \hat{p}^j} + (t-t_0)\delta_j^i \Big| \lesssim \epsilon t \tag{8.10}$$

Taking the determinant and ϵ small enough we have

$$\Big| det\Big(\frac{\partial y^i}{\partial \hat{p}^j}\Big) + (t-t_0)^3 \Big| \lesssim \epsilon t^3$$

Since $t \geq t_0+1$, (8.5) follows.

For the bound (8.6) we need estimates on $\frac{\partial \hat{p}^k}{\partial y^j}$. Indeed we have

$$\frac{\partial z_1^i(s,t)}{\partial y^j} = \frac{\partial X^i(s,t,x,\hat{p}(t,x,y))}{\partial y^j} = \frac{\partial X^i}{\partial \hat{p}^k}\frac{\partial \hat{p}^k}{\partial y^j} \tag{8.11}$$

In order to do that, first notice that the matrix $\left(\frac{\partial \hat{p}^i}{\partial y^j}\right)$ is the inverse of the matrix $\left(\frac{\partial y^i}{\partial \hat{p}^j}\right)$ and from (8.10) we get

$$\Big|\frac{\partial \hat{p}^i}{\partial y^j} + \frac{\delta^i_j}{t-t_0}\Big| \lesssim \frac{\epsilon}{t}$$

From

$$0 = \frac{\partial \hat{p}^i(t,x,y(t,x,\hat{p}))}{\partial x^j} = \frac{\partial \hat{p}^i}{\partial x^j}(t,x,y) + \frac{\partial \hat{p}^i}{\partial y^k}(t,x,y)\frac{\partial y^k}{\partial x^j}$$

we get

$$\frac{\partial \hat{p}^i}{\partial x^j} = -\frac{\partial \hat{p}^i}{\partial y^k}\frac{\partial y^k}{\partial x^j}$$

and setting $s = t_0$ in (8.3) gives

$$\Big|\frac{\partial y^i}{\partial x^j} - \delta^i_j\Big| \lesssim \epsilon$$

Now, since

$$\frac{\partial \hat{p}^i}{\partial x^j} - \frac{\delta^i_j}{t-t_0} = -\Big(\frac{\partial \hat{p}^i}{\partial y^k} + \frac{\delta^k_j}{t-t_0}\Big)\big(\frac{\partial y^k}{\partial x^j} - \delta^k_j\big) + \frac{1}{t-t_0}\big(\frac{\partial y^i}{\partial x^j} - \delta^i_j\big) - \Big(\frac{\partial \hat{p}^i}{\partial y^j} + \frac{\delta^i_j}{t-t_0}\Big)$$

Hence, with $t \geq t_0 + 1$

$$\Big|\frac{\partial \hat{p}^i}{\partial x^j} - \frac{\delta^i_j}{t-t_0}\Big| \lesssim \frac{\epsilon}{t}$$

With (8.11) we get

$$\begin{aligned}&\Big|\frac{\partial z_1^i}{\partial y^j} - \frac{t-s}{t-t_0}\delta^i_j\Big| \\ &\quad= \Big|\Big(\frac{\partial X^i}{\partial \hat{p}^k} + (t-s)\delta^i_k\Big)\Big(\frac{\partial \hat{p}^k}{\partial y^j} - \frac{\delta^k_j}{t-t_0}\Big) - (t-s)\Big(\frac{\partial \hat{p}^i}{\partial y^j} - \frac{\delta^i_j}{t-t_0}\Big) + \frac{1}{t-t_0}\Big(\frac{\partial X^i}{\partial \hat{p}^j} + (t-s)\delta^i_j\Big)\Big| \lesssim \epsilon.\end{aligned}$$

Hence

$$\Big|\det\Big(\frac{\partial z_1^i}{\partial y^j}\Big) - \Big(\frac{t-s}{t-t_0}\Big)^3\Big| \lesssim \epsilon$$

and (5.6) follow taking ϵ sufficiently small for $t_0 \leq s \leq t_0 + \frac{1}{2}$.
The bound (8.8) follows with the same idea. We shall prove now the bound (8.9).
Recall that

$$\kappa = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial \hat{p}} \\ \frac{\partial \hat{P}}{\partial x} & \frac{\partial \hat{P}}{\partial \hat{p}} \end{pmatrix}$$

If we denote for $t_0 \leq s \leq t : W(s) = (X(s,t,x,\hat{p}), \hat{P}(s,t,x,\hat{p}))$ and $w = (x,\hat{p})$ we have $\kappa = \frac{dW}{dw}$
Now

$$\frac{d}{ds}(W(s)) = (\hat{P}(s,t,x,\hat{p}), \hat{P}^i\hat{P}^\alpha\hat{P}^\beta\Gamma^0_{\alpha\beta}(s,X) - \hat{P}^\alpha\hat{P}^\beta\Gamma^i_{\alpha\beta}(s,X))$$

We denote $F(s,W) = (\hat{P}(s,t,x,\hat{p}), \hat{P}^i\hat{P}^\alpha\hat{P}^\beta\Gamma^0_{\alpha\beta}(s,X) - \hat{P}^\alpha\hat{P}^\beta\Gamma^i_{\alpha\beta}(s,X))$
Moreover

$$\begin{aligned}\frac{d}{ds}\det\kappa(s) &= \mathrm{tr}\Big({}^tCom(\kappa)\frac{d\kappa}{ds}\Big) \\ &= \mathrm{tr}\Big(\kappa^{-1}\frac{d\kappa}{ds}(s)\Big)\cdot\det\kappa(s)\end{aligned}$$

We have

$$
\begin{aligned}
\operatorname{tr}\Big(\kappa^{-1}\frac{d\kappa}{ds}\Big) &= \sum_{i,j}[\kappa^{-1}]_{i,j}\Big[\frac{d\kappa}{ds}\Big]_{j,i} \\
&= \frac{\partial w^j}{\partial W^i}\frac{\partial F^i}{\partial w^j} \\
&= \frac{\partial F^i}{\partial W^i}
\end{aligned}
$$

and

$$
\frac{\partial F^i}{\partial W^i} = 3\hat{P}^\alpha \hat{P}^\beta \Gamma^0_{\alpha\beta}(s,X) + 2\hat{P}^i\hat{P}^\beta\Gamma^0_{i\beta}(s,X) - 2\hat{P}^\beta\Gamma^i_{i\beta}(s,X),
$$

Now with bounds (6.1) :

$$
\Big|\frac{d}{ds}\det\Big(\frac{\partial W}{\partial w}(s)\Big)\Big| \leq \frac{C\epsilon}{s^{1+a}}\Big|\det\Big(\frac{\partial W}{\partial w}(s)\Big)\Big|
$$

Using the Grönwall inequality, the bound (8.9) follows.

□

8.3 L^1 and L^2 estimates of components of the energy momentum tensor

One of the key inequality to prove in the paper of Claude Bardos and Pierre degond was

$$
\|\rho_n(.,t)\|_\infty + \|\nabla_x\rho_n(.,t)\|_\infty + \|\nabla^2_x\rho_n(.,t)\|_\infty \leq \frac{A}{(1+t)^3}, \quad \forall t>0
$$
$$
\|\rho_n(.,t)\|_1 + \|\nabla_x\rho_n(.,t)\|_1 \leq A\epsilon
$$

where

$$
\rho_n(t,x) = \sum_\alpha q_\alpha \int_{\mathbb{R}^3} u^n_\alpha(x,v,t)dv
$$

ρ_n could be assimilated as our stress-energy tensor in our case. Before showing the theorem 1.3 we have the following Lemma which can easily be shown using the Vlasov equation with characteristics, the change of variables $\hat{p} \mapsto y = X(t_0)$ as in [2] and the bound (8.5) :

The proof of the following lemmas and propositions until section 9 will be taken from [1] page 46 to 51 without too many changes since they are very well explained. Nevertheless, I chose to rewrite them for completeness purposes.

Lemma 8.2. *Suppose $\pi(supp(f_0)) \subset \{|x| \leq K\}$, $t \geq t_0+1$ and (6.1) holds. Then*

$$
\int \chi_{supp(f)}(t,x,\hat{p})d\hat{p} \lesssim \frac{1}{t^3},
$$

where $\chi_{supp(f)}$ is the characteristic function of $supp(f)$

Proposition 8.7. *Suppose $\pi(supp(f)) \subset \{|x| \leq ct\}$, $t \geq t_0+1$ and (6.1) holds. For ϵ sufficiently small then, for any multi index I with $|I| \leq N-1$ and each $\mu,\nu = 0,1,2,3$,*

$$
\|(Z^I T^{\mu\nu})(t,.)\|_{L^2} \leq \frac{C\mathcal{V}_I}{t^{3/2}} + C\mathbb{D}_{\lfloor\frac{|I|}{2}\rfloor+1}\Big(\sum_{|J|\leq|I|-1}\frac{\|(Z^J\Gamma)(t,.)\|_{L^2}}{t^{1+a}} + \sum_{|J|\leq|I|+1}\frac{1}{t^{3/2}}\int_{t_0}^t\frac{\|((Z^J\Gamma)(s,.)\|_{L^2}}{s^{a+1/2}}ds\Big),
$$

$$
\|(Z^I T^{\mu\nu})(t,.)\|_{L^1} \leq C\mathcal{V}_I + C\mathbb{D}_{\lfloor\frac{|I|}{2}\rfloor+1}\Big(\sum_{|J|\leq|I|-1}\frac{\|(Z^J\Gamma)(t,.)\|_{L^2}}{t^{a-1/2}} + \sum_{|J|\leq|I|+1}\int_{t_0}^t\frac{\|((Z^J\Gamma)(s,.)\|_{L^2}}{s^{a+1/2}}ds\Big),
$$

where we recall that

$$
\mathbb{D}_K = \sum_{k+l\leq K}\|\partial^k_x\partial^l_p f_0\|_{L^\infty} \qquad \mathcal{V}_K = \sum_{k+l\leq K}\|\partial^k_x\partial^l_p f_0\|_{L^2_xL^2_p}
$$

Proof. Recall that in the proposition 8.5 we provided some bounds on the vector fields applied to the energy tensor with derivatives of f.

In order to prove this result we shall use the change of variables described in the previous propositions and we shall try to bound in norme L^1 : $\|Z^I T^{\mu\nu}(t,.)F(t,.)\|_{L^1}$ for any function $F(t,x)$. Setting then $F(t,x)$ we will have our L^2 estimate and with $F(t,x) = \mathrm{sign}(Z^I T^{\mu\nu}(t,x))\chi_{\{|x|\le ct+K\}}$ the L^1 estimate will follow.

Let F be a function of (t,x). From Proposition 8.5 we get

$$\begin{aligned}\|Z^I T^{\mu\nu}(t,.)F(t,.)\|_{L^1} \le C &\sum_{n_1+n_2\le|I|} \int_{|x|\le ct+K}\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p}|F(t,x)|dx \\ &+ C\sum_{n_1+n_2\le\lfloor\frac{|I|}{2}\rfloor+1}\int_{|x|\le ct+K}\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0,X(t_0),\hat{P}(t_0))|\end{aligned}$$
$$\times\Big(t^{2-a}\sum_{|J|\le|I|-1}|(Z^J\Gamma)(t,x)| + \sum_{|J|\le|I|}\int_{t_0}^t (s)^{1-a}\Big|(Z^J\Gamma)(s,s\frac{x}{t})\Big| + s\Big|(Z^J\Gamma)(s,s\frac{x}{t}) - (Z^J\Gamma)(s,X(s))\Big|ds\Big)d\hat{p}|F(t,x)|dx.$$

With Cauchy-Schwarz inequality :

$$\begin{aligned}\int_{|x|\le ct+K}&\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p}|F(t,x)|dx \\ &\le \int\Big(\int \chi^2_{\mathrm{supp}(f)}(t,x,\hat{p})d\hat{p}\Big)^{1/2}\Big(\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0,X(t_0),\hat{P}(t_0))|^2 d\hat{p}\Big)^{1/2}|F(t,x)|dx\end{aligned}$$

for $n_1+n_2\le|I|$. From lemma 8.2 and applying again Cauchy-Schwarz inequality we get

$$\begin{aligned}\int_{|x|\le ct+K}&\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p}|F(t,x)|dx \\ &\le \int\Big(\int \chi^2_{\mathrm{supp}(f)}(t,x,\hat{p})d\hat{p}\Big)^{1/2}\Big(\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0,X(t_0),\hat{P}(t_0))|^2 d\hat{p}\Big)^{1/2}|F(t,x)|dx \\ &\le \frac{C}{t^{3/2}}\Big(\int\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0,X(t_0),\hat{P}(t_0))|^2 d\hat{p}dx\Big)^{1/2}\|F(t,.)\|_{L^2} \le \frac{C\mathcal{V}_{n_1+n_2}}{t^{3/2}}\|F(t,.)\|_{L^2}\end{aligned}$$

Similarly, for any multi index J, using the bound (8.5) we have :

$$\begin{aligned}&\int_{|x|\le ct+K}\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p}t^{2-a}|(Z^J\Gamma)(t,x)||F(t,x)|dx \\ &\le \int_{|x|\le ct+K}\int_{|y|\le K}|(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0,y,\hat{P}(t_0,t,x,\hat{p}(t,x,y)))|\Big|\det\Big(\frac{\partial\hat{p}^i}{\partial y^j}\Big)\Big|dyt^{2-a}|(Z^J\Gamma)(t,x)||F(t,x)|dx \\ &\le \frac{C\mathbb{D}_{n_1+n_2}}{t^{1+a}}\|(Z^J\Gamma)(t,.)\|_{L^2}\|F(t,.)\|_{L^2}\end{aligned}$$

For the third term of the bound of $\|Z^I T^{\mu\nu}(t,.)F(t,.)\|_{L^1}$ we use the change of variables $x^i\mapsto z^i := s\frac{x^i}{t}$ and we have :

$$\begin{aligned}\int_{|x|\le ct+K}&\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p}\int_{t_0}^t s^{1-a}\Big|(Z^J\Gamma)(s,s\frac{x}{t})\Big|ds|F(t,x)|dx \\ &\le \frac{C\mathbb{D}_{n_1+n_2}}{t^3}\int_{t_0}^t s^{1-a}\Big(\int_{|x|\le ct+K}\Big|(Z^J\Gamma)(s,s\frac{x}{t})\Big|^2 dx\Big)^{1/2}\|F(t,.)\|_{L^2}ds \\ &\le \frac{C\mathbb{D}_{n_1+n_2}}{t^{3/2}}\int_{t_0}^t \frac{\|(Z^J\Gamma)(s,.)\|_{L^2}}{s^{a+1/2}}ds\|F(t,.)\|_{L^2}\end{aligned}$$

Now we need to bound the last term which is

$$\int_{|x|\le ct+K}\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2} f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p}\int_{t_0}^t s\Big|(Z^J\Gamma)(s,s\frac{x}{t}) - (Z^J\Gamma)(s,X(s))\Big|ds|F(t,x)|dx$$

In order to do that we need to recall the change of variables we gave in proposition 8.6 which can be used depending on whether $t_0+\frac{1}{2}\leq s\leq t$ or not. Therefore we need to divide the last term in two parts :

$$\sum_{|J|\leq|I|}\int_{t_0}^{t} s\Big|(Z^J\Gamma)(s,s\frac{x}{t})-(Z^J\Gamma)(s,X(s))\Big|ds$$
$$\leq\sum_{|J|\leq|I|}\int_{t_0}^{t_0+1/2} s\Big|(Z^J\Gamma)(s,s\frac{x}{t})\Big|+s\Big|(Z^J\Gamma)(s,X(s))\Big|ds+\sum_{|J|\leq|I|}\int_{t_0+1/2}^{t} s\Big|(Z^J\Gamma)(s,s\frac{x}{t})-(Z^J\Gamma)(s,X(s))\Big|ds$$

The first term can be estimated as above and we get

$$\int_{|x|\leq ct+K}\int|(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p}\int_{t_0}^{t_0+1/2} s\Big|(Z^J\Gamma)(s,s\frac{x}{t})\Big|ds|F(t,x)|dx$$
$$\leq\frac{C\mathbb{D}_{n_1+n_2}}{t^{3/2}}\int_{t_0}^{t_0+1/2}\|(Z^J\Gamma)(s,.)\|_{L^2}ds\|F(t,.)\|_{L^2}$$
$$\leq\frac{C\mathbb{D}_{n_1+n_2}}{t^{3/2}}\int_{t_0}^{t}\frac{\|(Z^J\Gamma)(s,.)\|_{L^2}}{s^{1/2+a}}ds\|F(t,.)\|_{L^2}$$

For the second term we have :

$$\int_{|x|\leq ct+K}\int|(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))|\int_{t_0}^{t_0+1/2} s\Big|(Z^J\Gamma)(s,X(s))\Big|dsd\hat{p}|F(t,x)|dx$$
$$\leq\frac{C\mathbb{D}_{n_1+n_2}}{t^{3}}\int_{t_0}^{t_0+1/2}\int_{|x|\leq ct+K}|F(t,x)|\int_{|y|\leq K}\Big|(Z^J\Gamma)(s,X(s))\Big|dydxds$$

We use the change of variables associated to the bound (8.6) from Proposition 8.6. And we get

$$\frac{C\mathbb{D}_{n_1+n_2}}{t^{3}}\int_{t_0}^{t_0+1/2}\int_{|x|\leq ct+K}|F(t,x)|\Big(\int_{|y|\leq K}\Big|(Z^J\Gamma)(s,X(s))\Big|^2dy\Big)^{1/2}dxds$$
$$\leq\frac{C\mathbb{D}_{n_1+n_2}}{t^{3}}\int_{t_0}^{t_0+1/2}\|(Z^J\Gamma)(s,.)\|_{L^2}ds\|F(t,.)\|_{L^2}$$

Finally we have

$$\int_{|x|\leq ct+K}\int|(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))|\int_{t_0}^{t_0+1/2} s\Big|(Z^J\Gamma)(s,X(s))\Big|dsd\hat{p}|F(t,x)|dx$$
$$\leq\frac{C\mathbb{D}_{n_1+n_2}}{t^{3/2}}\int_{t_0}^{t}\frac{\|(Z^J\Gamma)(s,.)\|_{L^2}}{s^{1/2+a}}ds\|F(t,.)\|_{L^2}$$

For the last term we write :

$$(Z^J\Gamma)(s,s\frac{x}{t})-(Z^J\Gamma)(s,X(s))=\int_0^1\frac{d}{d\sigma}\Big(Z^J\Gamma\big(s,\sigma s\frac{x}{t}+(1-\sigma)X(s)\big)\Big)d\sigma$$

Moreover we have

$$\frac{d}{d\sigma}\Big(Z^J\Gamma\big(s,\sigma s\frac{x}{t}+(1-\sigma)X(s)\big)\Big)=\Big(s\frac{x^l}{t}-X(s)^l\Big)(\partial_{x^l}Z^J\Gamma)\Big(s,\sigma s\frac{x}{t}+(1-\sigma)X(s)\Big)$$

Hence

$$(Z^J\Gamma)(s,s\frac{x}{t})-(Z^J\Gamma)(s,X(s))=\Big(s\frac{x^l}{t}-X(s)^l\Big)\int_0^1(\partial_{x^l}Z^J\Gamma)\Big(s,\sigma s\frac{x}{t}+(1-\sigma)X(s)\Big)d\sigma$$

Now with Proposition 4.3 and (6.14) we get

$$\Big|s\frac{x^l}{t}-X(s)^l\Big|\leq Cs^{1-a}$$

Hence we have

$$\sum_{|J|\le|I|} s\Big|(Z^J\Gamma)(s,s\frac{x}{t})-(Z^J\Gamma)(s,X(s))\Big|ds \le C\sum_{|J|\le|I|+1} s^{1-a}\int_0^1 \Big|(Z^J\Gamma)\big(s,\sigma s\frac{x}{t}+(1-\sigma)X(s)\big)\Big|d\sigma$$

where the fact that

$$\partial^k\Gamma(t,x)=t^{-k}\sum_{|K|\le k}A_K(x/t)(Z^K\Gamma)(t,x)$$

has been used with functions A smooth when $|x|/t\le 1$

Recall that if $t_0+\frac{1}{2}\le s\le t$ and $0\le\sigma\le 1$, then the change of variables $(x,y)\mapsto(z_2(\sigma,s,t,x,y),y)$ satisfies

$$\Big|det\Big(\frac{\partial z_2^i(\sigma,s,t)}{\partial x^j}\Big)^{-1}\Big|\le C\Big(\frac{t}{s}\Big)^3$$

where

$$z_2^i(\sigma,s,t,x,y):=\sigma s\frac{x^i}{t}+(1-\sigma)X(s,t,x,\hat{p}(t,x,y))^i,$$

We now have

$$\int_{|x|\le ct+K}\int|(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))|\sum_{|J|\le|I|}\int_{t_0+1/2}^t s\Big|(Z^J\Gamma)(s,s\frac{x}{t})-(Z^J\Gamma)(s,X(s))\Big|dsd\hat{p}|F(t,x)|dx$$

$$\le\frac{C\mathbb{D}_{n_1+n_2}}{t^3}C\sum_{|J|\le|I|+1}\int_{|x|\le ct+K}\int_{|y|\le K}\int_{t_0+1/2}^t s^{1-a}\int_0^1\Big|(Z^J\Gamma)\big(s,\sigma s\frac{x}{t}+(1-\sigma)X(s)\big)\Big|d\sigma dsdy|F(t,x)|dx$$

$$\le\frac{C\mathbb{D}_{n_1+n_2}}{t^3}C\sum_{|J|\le|I|+1}\int_{t_0+1/2}^t s^{1-a}\int_0^1\int_{|y|\le K}\Big(\int_{|x|\le ct+K}\Big|(Z^J\Gamma)\big(s,\sigma s\frac{x}{t}+(1-\sigma)X(s)\big)\Big|^2dx\Big)^{1/2}dyd\sigma ds\|F(t,x)\|_{L^2}$$

$$\le\frac{C\mathbb{D}_{n_1+n_2}}{t^{3/2}}\sum_{|J|\le|I|+1}\int_{t_0}^t\frac{\|(Z^J\Gamma)(s,.)\|_{L^2}}{s^{1/2+a}}ds\|F(t,.)\|_{L^2}$$

Combining every previous results we get

$$\|Z^IT^{\mu\nu}(t,.)F(t,.)\|_{L^1}\le\frac{C\mathcal{V}_{|I|}\|F(t,.)\|_{L^2}}{t^{3/2}}$$
$$+C\mathbb{D}_{\lfloor\frac{|I|}{2}\rfloor+1}\|F(t,.)\|_{L^2}\Big(\sum_{|J|\le|I|-1}\frac{\|(Z^J\Gamma)(t,.)\|_{L^2}}{t^{1+a}}+\sum_{|J|\le|I|+1}\frac{1}{t^{3/2}}\int_{t_0}^t\frac{\|((Z^J\Gamma)(s,.)\|_{L^2}}{s^{a+1/2}}ds\Big)$$

Now, setting then $F(t,x)$ we will have our L^2 estimate and with $F(t,x)=\mathrm{sign}(Z^IT^{\mu\nu}(t,x))\chi_{\{|x|\le ct+K\}}$ the L^1 estimate will follow using also the fact that $\mathrm{supp}(T^{\mu\nu})\in\{|x|\le ct+K\}$ and $\|\chi_{\{|x|\le ct+K\}}\|_{L^2}\le Ct^{3/2}$.

□

Proposition 8.8. *Suppose $\pi(supp(f))\subset\{|x|\le ct\}$, $t\ge t_0+1$ and (6.1) holds. For ϵ sufficiently small then, for any multi index I with $|I|\le N$ and each $\mu,\nu=0,1,2,3$,*

$$\|(Z^IT^{\mu\nu})(t,.)\|_{L^2}\le\frac{C\mathcal{V}_I}{t^{3/2}}+C\mathbb{D}_{\lfloor\frac{|I|}{2}\rfloor+1}\Big(\sum_{|J|\le|I|-1}\frac{\|(Z^J\Gamma)(t,.)\|_{L^2}}{t^{1+a}}+\sum_{|J|\le|I|}\frac{1}{t^{3/2}}\int_{t_0}^t\frac{\|((Z^J\Gamma)(s,.)\|_{L^2}}{s^{a+1/2}}ds\Big)$$

Proof. As we can see the previous proposition has been modified with the last sum only providing $|J|\le|I|$ and with a power of s slightly worse. Since we got this sum in the last proof of the theorem from the "last term", we only need to change this last part. Combining previous result of the last proof we can show that

$$
\begin{aligned}
\|Z^I T^{\mu\nu}(t,.)F(t,.)\|_{L^1} \leq \; & \frac{C\mathcal{V}_{|I|}\|F(t,.)\|_{L^2}}{t^{3/2}} \\
& + C\mathbb{D}_{\lfloor \frac{|I|}{2} \rfloor+1}\|F(t,.)\|_{L^2}\Big(\sum_{|J|\leq|I|-1} \frac{\|(Z^J\Gamma)(t,.)\|_{L^2}}{t^{1+a}} + \sum_{|J|\leq|I|} \frac{1}{t^{3/2}}\int_{t_0}^t \frac{\|((Z^J\Gamma)(s,.)\|_{L^2}}{s^{a+1/2}}ds\Big) \\
& + C\sum_{n_1+n_2\leq\lfloor\frac{|I|}{2}\rfloor+1}\ \sum_{|J|\leq|I|}\int_{|x|\leq ct+K}\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))| \\
& \times \int_{t_0+1/2}^t s\Big|(Z^J\Gamma)(s,s\frac{x}{t}) - (Z^J\Gamma)(s,X(s))\Big| ds d\hat{p}|F(t,x)|dx
\end{aligned}
$$

And we shall bound only the last term differently.
Now we have

$$
\sum_{|J|\leq|I|}\int_{t_0+1/2}^t s\Big|(Z^J\Gamma)(s,s\frac{x}{t}) - (Z^J\Gamma)(s,X(s))\Big|ds \leq \sum_{|J|\leq|I|}\int_{t_0+1/2}^t s\Big|(Z^J\Gamma)(s,s\frac{x}{t})\Big| + \Big|(Z^J\Gamma)(s,X(s))\Big|ds
$$

Recall that in the previous proof we have shown that

$$
\begin{aligned}
&\int_{|x|\leq ct+K}\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p}\int_{t_0}^t s^{1-a}\Big|(Z^J\Gamma)(s,s\frac{x}{t})\Big|ds|F(t,x)|dx \\
&\qquad \leq \frac{C\mathbb{D}_{n_1+n_2}}{t^3}\int_{t_0}^t s^{1-a}\Big(\int_{|x|\leq ct+K}\Big|(Z^J\Gamma)(s,s\frac{x}{t})\Big|^2dx\Big)^{1/2}\|F(t,.)\|_{L^2}ds \\
&\qquad\qquad \leq \frac{C\mathbb{D}_{n_1+n_2}}{t^{3/2}}\int_{t_0}^t \frac{\|(Z^J\Gamma)(s,.)\|_{L^2}}{s^{a+1/2}}ds\|F(t,.)\|_{L^2}
\end{aligned}
$$

where we used the change of variables $x^i \mapsto z^i := s\frac{x^i}{t}$. Now we have the same estimate to do but with $a=0$ which leads to

$$
\begin{aligned}
&\int_{|x|\leq ct+K}\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p}\int_{t_0+1/2}^t s\Big|(Z^J\Gamma)(s,s\frac{x}{t})\Big|dsd\hat{p}|F(t,x)|dx \\
&\qquad \leq \frac{C\mathbb{D}_{n_1+n_2}}{t^3}\int_{t_0+1/2}^t s\Big(\int_{|x|\leq ct+K}\Big|(Z^J\Gamma)(s,s\frac{x}{t})\Big|^2dx\Big)^{1/2}\|F(t,.)\|_{L^2}ds \\
&\qquad\qquad \leq \frac{C\mathbb{D}_{n_1+n_2}}{t^{3/2}}\int_{t_0}^t \frac{\|(Z^J\Gamma)(s,.)\|_{L^2}}{s^{1/2}}ds\|F(t,.)\|_{L^2}
\end{aligned}
$$

For the second term recall that if $t_0+\frac{1}{2}\leq s\leq t$ and $0\leq\sigma\leq 1$, then the change of variables $(x,y)\mapsto(z_2(\sigma,s,t,x,y),y)$ satisfies

$$
\Big|det\Big(\frac{\partial z_2^i(\sigma,s,t)}{\partial x^j}\Big)^{-1}\Big| \leq C\Big(\frac{t}{s}\Big)^3
$$

where

$$
z_2^i(\sigma,s,t,x,y) := \sigma s\frac{x^i}{t} + (1-\sigma)X(s,t,x,\hat{p}(t,x,y))^i,
$$

Hence

$$
\begin{aligned}
&\int_{|x|\leq ct+K}\int |(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p}\int_{t_0+1/2}^t s\Big|(Z^J\Gamma)(s,X(s))\Big|dsd\hat{p}|F(t,x)|dx \\
&\qquad \leq \frac{C\mathbb{D}_{n_1+n_2}}{t^3}\int_{t_0+1/2}^t s\int_{|y|\leq K}\Big(\int_{|x|\leq ct+K}\Big|(Z^J\Gamma)(s,X(s))\Big|^2dx\Big)^{1/2}dyds\|F(t,.)\|_{L^2} \\
&\qquad\qquad \leq \frac{C\mathbb{D}_{n_1+n_2}}{t^{3/2}}\int_{t_0}^t \frac{\|(Z^J\Gamma)(s,.)\|_{L^2}}{s^{1/2}}ds\|F(t,.)\|_{L^2}
\end{aligned}
$$

Finally we get

$$\|Z^I T^{\mu\nu}(t,.)F(t,.)\|_{L^1} \leq \frac{C\mathcal{V}_{|I|}\|F(t,.)\|_{L^2}}{t^{3/2}}$$
$$+ C\mathbb{D}_{\lfloor\frac{|I|}{2}\rfloor+1}\|F(t,.)\|_{L^2}\Big(\sum_{|J|\leq|I|-1}\frac{\|(Z^J\Gamma)(t,.)\|_{L^2}}{t^{1+a}} + \sum_{|J|\leq|I|}\frac{1}{t^{3/2}}\int_{t_0}^{t}\frac{\|((Z^J\Gamma)(s,.)\|_{L^2}}{s^{1/2}}ds\Big)$$

and the proof follows by setting $F(t,x) = Z^I T^{\mu\nu}$

□

8.4 Proof of Theorem 3.2

Now that we have shown the case $t \geq t_0 + 1$ we need to consider the case $t_0 \leq t \leq t_0 + 1$

Proposition 8.9. *Suppose $\pi(supp(f)) \subset \{|x| \leq ct\}$, $t_0 \leq t \leq t_0 + 1$ and (6.1) holds. For ϵ sufficiently small then, for any multi index I with $|I| \leq N$ and each $\mu, \nu = 0, 1, 2, 3$,*

$$\|(\partial^I T^{\mu\nu})(t,.)\|_{L^2}+\|(\partial^I T^{\mu\nu})(t,.)\|_{L^1} \leq \mathcal{V}_I+\mathbb{D}_{\lfloor\frac{|I|}{2}\rfloor+1}\Big(\sum_{|J|\leq|I|-1}\|(\partial^J\Gamma)(t,.)\|_{L^2}+\sum_{|J|\leq|I|}\int_{t_0}^{t}\|((\partial^J\Gamma)(s,.)\|_{L^2}ds\Big)$$

Proof. We can show that as in the proposition 8.5 and with the help of proposition 6.15 and 6.16 :

$$|\partial^I T^{\mu\nu}(t,x)| \lesssim \sum_{n_1+n_2\leq|I|}\int|(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p}+\sum_{n_1+n_2\leq\lfloor\frac{|I|}{2}\rfloor+1}\int|(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))|$$
$$\times\Big(\sum_{|J|\leq|I|-1}|(\partial^J\Gamma)(t,x)| + \sum_{|J|\leq|I|}\int_{t_0}^{t}\Big|(\partial^J\Gamma)(s,X(s))\Big|ds\Big)d\hat{p}$$
$$\lesssim \sum_{n_1+n_2\leq|I|}\int|(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p}+\mathbb{D}_{\lfloor\frac{|I|}{2}\rfloor+1}\Big(\sum_{|J|\leq|I|-1}|(\partial^J\Gamma)(t,x)|+\sum_{|J|\leq|I|}\int_{t_0}^{t}\Big|(\partial^J\Gamma)(s,X(s))\Big|ds\Big)d\hat{p}$$

Moreover fon any function F(t,x),

$$\int\int|(\partial_x^{n_1}\partial_{\hat{p}}^{n_2}f)(t_0,X(t_0),\hat{P}(t_0))|d\hat{p}|F(t,x)|dx \lesssim \mathcal{V}_{n_1+n_2}\|F(t,.)\|_{L^2}$$

and

$$\int|F(t,x)|\int\int_{t_0}^{t}\big|(\partial^J\Gamma)(s,X(s))\big|dsd\hat{p}dx \lesssim \int_{t_0}^{t}\|(\partial^J\Gamma)(s,.)\|_{L^2}ds\|F(t,.)\|_{L^2}$$

where the change of variables $x \mapsto X(s,t,x,\hat{p})$ and the bound (8.3) have been used.
Combining every results we get

$$\|\partial^I T^{\mu\nu}(t,.)F(t,.)\|_{L^1} \lesssim \mathcal{V}_I\|F(t,.)\|_{L^2}$$
$$+\mathbb{D}_{\lfloor\frac{|I|}{2}\rfloor+1}\int|F(t,x)|\Big(\sum_{|J|\leq|I|-1}|(\partial^J\Gamma)(t,.)| + \sum_{|J|\leq|I|}\int\int_{t_0}^{t}|(\partial^J\Gamma)(s,X(s))|dsd\hat{p}\Big)dx$$
$$\lesssim \|F(t,.)\|_{L^2}\Big[\mathcal{V}_I + \mathbb{D}_{\lfloor\frac{|I|}{2}\rfloor+1}\Big(\sum_{|J|\leq|I|-1}\|(\partial^J\Gamma)(t,.)\|_{L^2} + \sum_{|J|\leq|I|}\int_{t_0}^{t}\|(\partial^J\Gamma)(s,.)\|_{L^2}ds\Big)\Big]$$

and setting $F = \partial^I T^{\mu\nu}$ first and then $F = sign(\partial^I T^{\mu\nu})$ gives the result.

□

We now have all the ingredients to prove theorem 3.2. From proposition 4.1 we have that $|x| \leq ct + K$ for $t \geq 0$ in supp($T^{\mu\nu}$) which is $|x| \leq c\tilde{t}$ for $\tilde{t} \geq t_0$ where $t_0 := K/c$. Hence $\tilde{t} = t_0 + t$. We note that $\tilde{\Omega}_{ij} = \Omega_{ij}$, $\tilde{B}_i = \tilde{t}\partial_{x^i} + x^i\partial_t$, $\tilde{S} = \tilde{t}\partial_t + x^k\partial_{x^k}$. These vector fields satisfy for any multi index I the following property :

$$\tilde{t}^{|I|}\partial^I = \sum_{|J|\leq|I|}\Lambda_{IJ}(\frac{x}{\tilde{t}})\tilde{Z}^J \qquad \text{if } |x|/\tilde{t} \leq c < 1 \tag{8.12}$$

for some smooth functions Λ_{IJ}.
Now we have for any function F and $t_0 \leq t \leq t_0 + 1$ and $|x| \leq ct$

$$\sum_{|J| \leq |I|} |\partial^I F(t,x)| \lesssim |Z^I F(t,x)| \lesssim \sum_{|J| \leq |I|} |\partial^I F(t,x)|$$

Proposition 8.7 and 8.8 hold for $t \geq t_0$ and we have from (8.12) that for $t \geq t_0$

$$\|\partial^I Z^J T^{\mu\nu}(t,.)\|_{L^1} \lesssim \sum_{|K| \leq |I|+|J|} \|Z^K T^{\mu\nu}(t,.)\|_{L^1}$$
$$\|\partial^I Z^J T^{\mu\nu}(t,.)\|_{L^2} \lesssim \sum_{|K| \leq |I|+|J|} \|Z^K T^{\mu\nu}(t,.)\|_{L^2}$$

Hence the spacetime derivatives ∂^I can be included in proposition 8.7 and 8.8. If we suppose the assumptions of theorem 3.2, then the bounds hold with t replaced by $\tilde{t}$ and the vector field Z replaced by $\tilde{Z}$ and the proof of theorem 3.2 follows.

9 The Einstein equations

In this section we provide the main results in order to show the theorem 3.1. Recall that in order to prove theorem 3.1 one gives T_* the supremum of all times T_1 such that a solution of the reduced Einstein-Vlasov system attaining the given data exists for all $t \in [0, T_1]$ and satisfies the bounds :

(9.1) $$E_N(t)^{1/2} \leq C_N \epsilon (1+t)^\delta, \qquad \sum_{|I| \leq N-1} \|Z^I \hat{T}(t,.)\|_{L^1} \leq C_N \epsilon$$

for all $t \in [0, T_1]$, where

$$E_N(t) = \sum_{|I| \leq N} \|w^{1/2} \partial Z^I h^1(t,.)\|^2_{L^2}, \qquad w(t,x) = \begin{cases} (1+|r-t|)^{1+2\gamma} & r > t \\ 1 + (1+|r-t|)^{-2\mu} & r \leq t \end{cases}$$

and δ is such that $0 < 8\delta < \gamma < 1 - 8\delta$, $M \leq \epsilon$ and C_N is a fixed large constant, $N \geq 4$. The set of such T_1 is non empty by the local existence theorem of Choquet-Bruhat. The main objective is thus to use the Einstein equations to prove that the bounds in fact hold with better constants, provided the initial data are sufficiently small.

9.1 Klainerman-Sobolev inequality

In order to understand the importance of the Klainerman-Sobolev inequality, we first consider a semilinear equation in dimension $n \geq 4$

(9.2) $$\Box u = F(u, \partial u)$$

with initial conditions $u|_{t=0} = \epsilon f$ and $\partial_t u|_{t=0} = \epsilon g$ a. The question of the global existence of Cauchy problems requires good estimates on $|u(t,x)|$ for large t. The idea of the proof is to consider the energy

$$Q_n(t) = \sum_{|I| \leq n+4} \|\partial Z^I u(t)\|^2_{L^2}$$

Let $T* = \{T > 0 \,\text{such that we have a solution for our system}\, u \in C^\infty([0,T] \times \mathbb{R}^n)\}$
We suppose that Q verify :

$$Q_n(t) \leq A\epsilon, \qquad 0 \leq t \leq T*$$

where A is a constant
We now have the Klainerman-Sobolev inequality

Lemma 9.1. *Let $u \in C^\infty([0,T] \times \mathbb{R}^n)$ vanish when $|x|$ large. Then*

$$(1+t+|x|)^{n-1}(1+|t-|x||)|u(t,x)|^2 \leq C_n \sum_{|J| \leq \frac{n+2}{2}} \|Z^J u(t,.)\|^2_{L^2}$$

With this inequality and $Q_n(t) \leq A\epsilon$ one can show that for $|K| \leq (n+6)/2$

$$|\partial Z^K u(t,x)| \leq CA\epsilon(1+t)^{-\frac{n-1}{2}}$$

and then

$$Q_n(t) \leq \frac{1}{4} A\epsilon \, exp\Big[CA\epsilon \int_0^t \frac{d\tau}{(1+\tau)^{(n-1)/2}}\Big]$$

For $n \geq 4$ the integral is well defined and we can choose ϵ_0 small enough such that for every $0 < \epsilon < \epsilon_0$,

$$Q_n(t) \leq \frac{A\epsilon}{2}$$

and from a continuity argument the proof of the global existence of the solution follows.

In our case we consider the Klainerman-Sobolev inequality with weights which comes from a generalization of the Klainerman-Sobolev inequalities.

Lemma 9.2. *For an arbitrary function $\phi(t,x) \in C_0^\infty(\mathbb{R}^3)$ and an arbitrary (t,x)*

$$|\phi(t,x)|w^{1/2} \leq \frac{C\sum_{|I|\leq 2} \|w^{1/2} Z^I \phi(.,t)\|_{L^2}}{(1+t+r)(1+|t-r|)^{1/2}}$$

where

$$w(t,x) = \begin{cases} (1+|r-t|)^{1+2\gamma} & r > t \\ 1 + (1+|r-t|)^{-2\mu} & r \leq t \end{cases}$$

The proof of this inequality can be seen in [7] p. 68.

9.2 Weak L^∞ decay estimates

Recall that one of the decay estimates stated in theorem 3.1 was

$$|Z^I h^1(t,x)| \leq \frac{C'_N \epsilon (1+t)^{C'_N \epsilon}}{(1+t+r)(1+q_+)^\gamma}, \qquad |I| \leq N-3, \qquad q_+ = \begin{cases} r-t, & r > t \\ 0, & r \leq t \end{cases}$$

with the notations of theorem 3.1. In order to have estimates on the metric we use the Klainerman-Sobolev inequalities with weights (lemma 9.2). First we define the inverse of the metric

$$g^{\mu\nu} = m^{\mu\nu} + H^{\mu\nu} \qquad H^{\mu\nu} = H_0^{\mu\nu} + H_1^{\mu\nu} \qquad H_0^{\mu\nu} = -\chi\big(\frac{r}{1+t}\big)\frac{M}{r}\delta^{\mu\nu} \tag{9.3}$$

Proposition 9.1. *Suppose that the weak energy bounds (9.1) holds. Then*

$$|\partial Z^I h^1(t,x)| \leq \begin{cases} \frac{C\epsilon(1+t)^\delta}{(1+t+r)(1+|r-t|)^{1+\gamma}}, & r > t \\ \\ \frac{C\epsilon(1+t)^\delta}{(1+t+r)(1+|r-t|)^{1/2}}, & r < t \end{cases}, \qquad |I| \leq N-2 \tag{9.4}$$

$$|Z^I h^1(t,x)| \leq \begin{cases} \frac{C\epsilon(1+t)^\delta}{(1+t+r)(1+|r-t|)^{\gamma}}, & r > t \\ \\ \frac{C\epsilon(1+t)^\delta(1+|r-t|)^{1/2}}{(1+t+r)}, & r < t \end{cases}, \qquad |I| \leq N-2 \tag{9.5}$$

The same estimates hold for H_1 in place of h^1, and for h or H in place of h_1 if γ is replaced by δ.

Before beginning the proof of this property we need to provide a Lemma that will allow us to have the same estimates for H_1 instead of h^1 for nearly all the propositions.

Lemma 9.3. *Let $m^{\mu\nu}$ be the inverse of $m_{\mu\nu}$. Then, for small h :*

$$H^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu} = -h^{\mu\nu} + O^{\mu\nu}(h^2) \quad where \quad h^{\mu\nu} = m^{\mu\mu'} m^{\nu\nu'} h_{\mu'\nu'}$$

Proof. For simpleness purpose we write $g = m + h$ and $g^{-1} = m^{-1} + H$ without the index μ and ν. For h small we have

$$g^{-1} = (m+h)^{-1} = m^{-1} + m^{-1}\sum_{k=1}^{\infty}(-1)^k(hm^{-1})^k$$

Now

$$H = -hm^{-1} + O(h^2)$$

for h small and the lemma follows.

□

We can easily deduce from this lemma that

$$H_1^{\mu\nu} = -h^{1\mu\nu} + O(h^2)$$

Proof of Proposion 9.1. We first prove the inequality (9.4). Applying the Klainerman-Sobolev inequality (lemma 9.2) to $\phi(t,x) = \partial Z^I h^1(t,x)$ we get :

$$\begin{aligned}
|\partial Z^I h^1(t,x)|w^{1/2} &\le \frac{C\sum_{|J|\le N}\|w^{1/2}\partial Z^J h^1(t,.)\|_{L^2}}{(1+t+r)(1+|t-r|)^{1/2}} \\
&\le \frac{CNE_N(t)^{1/2}}{(1+t+r)(1+|t-r|)^{1/2}} \\
&\le \frac{C'_N\epsilon(1+t)^\delta}{(1+t+r)(1+|t-r|)^{1/2}}
\end{aligned}$$

Replacing w with its value gives (9.4) for h^1.

For (9.15) we first need to show that $\lim\limits_{|x|\to\infty} |Z^I h^1(0,x)| = 0$. The case $|I| = 0$ comes from the expression of h^1 at $t = 0$. For $|I| \ge 1$ the form of the vector fields provide $|Z\phi| \le C(1+t+|x|)|\partial\phi|$ for a function ϕ and (9.4) gives the result.
Hence for $t = 0$ the bound (9.5) follows from integrating (9.4) between 0 and ∞. For $t \ne 0$ we distinguish two cases :
For $r > t$: we consider $\omega = \frac{x}{|x|}$ fixed and we have

$$Z^I h^1(t,r\omega) = \int_{t+r}^{r} \partial Z^I h^1(t+r-\rho,\rho\omega)d\rho + Z^I h^1(0,(t+r)\omega)$$

Hence

$$\begin{aligned}
|Z^I h^1(t,r\omega)| &= \int_{r}^{t+r} |\partial Z^I h^1(t+r-\rho,\rho\omega)|d\rho + |Z^I h^1(0,(t+r)\omega)| \\
&\le \int_{r}^{\infty} \left|\frac{C\epsilon(1+t+r-\rho)^\delta}{(1+t+r)(1+|2\rho-t-r|)^{1+\gamma}}\right|d\rho + \left|\frac{C\epsilon}{(1+(t+r)\omega)(1+t+r)^\gamma}\right|
\end{aligned}$$

Bounding the integral by replacing ρ by r in the numerator and integrating between r and ∞ we get

$$|Z^I h^1(t,r\omega)| \le \frac{2\gamma C\epsilon(1+t)^\delta}{(1+t+r)(1+|r-t|)^\gamma} + \left|\frac{C\epsilon}{(1+t+r)(1+|r-t|)^\gamma}\right|$$

where the fact that $|r-t| \le t+r$ has been used

The bound (9.5) follows for $r > t$ and the same idea can be used for $r < t$.
In order to prove those bounds for h or H we need to prove them for h_0 then the previous bound on h^1 and the Lemma 9.3 will allow us to conclude. Recall that

$$h^0_{\mu\nu}(t,x) = \chi(\frac{r}{t})\chi(r)\frac{M}{r}\delta_{\mu\nu}$$

and χ is a smooth cut off function such that $0 \le \chi \le 1$,$\chi(s) = 1$ for $s \ge 3/4$ and $\chi(s) = 0$ for $s \le 1/2$

Applying Klainerman-Sobolev inequality with weights to h^0 we need to have some bounds on $Z^I h^0(t,.)$.
Since the vector fields admit at most one derivative with respect to t we have for $|I| \le 2$

$$|Z^I h^0_{\mu\nu}(t,.)| \le R\sum_{i=0}^{4}\sum_{j=0}^{2}\sum_{p=1}^{2}\frac{r^j}{t^i}\chi^{(p)}\left(\frac{r}{t}\right)\chi(r)\frac{M}{r}\delta_{\mu\nu}$$

where R is a constant. With the defintion of χ and we have to make the bounds only for t and r bounded hence we have

$$|Z^I h^0_{\mu\nu}(t,.)| \leq R'$$

and with Klainerman-Sobolev inequality with weights we get

$$|\partial Z^I h^0(t,x)| w^{1/2} \leq \frac{C''_N}{(1+t+r)(1+|t-r|)^{1/2}}$$

Replacing w by the expression give (9.4). Integrating as previously give (9.5) and the proposition follows.

□

In order to get improve decay estimates in the interior we use Hörmander $L^1 - L^\infty$ estimates which are the following ones :

Lemma 9.4. *Suppose that u is a solution of $\Box u = F$, where $\Box$ denotes the wave operator with data $u|_{t=0} = 0$ and $\partial_t u|_{t=0} = 0$, then :*

$$|u(t,x)|(1+t+|x|) \leq C \sum_{|I|\leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{|Z^I F(s,y)|}{1+s+|y|} dyds,$$

Lemma 9.5. *If v is the solution of $\Box v = 0$, with data $v|_{t=0} = v_0$ and $\partial_t v|_{t=0} = v_1$, then for any $\gamma > 0$;*

$$(1+t)|v(t,x)| \leq C \sup_x ((1+|x|)^{2+\gamma}(|v_1(x)| + |\partial v_0(x)|) + (1+|x|)^{1+\gamma}|v_0(x)|)$$

Moreover, we will need the following version of Hardy's inequality :

Lemma 9.6. *For any $-1 \leq a \leq 1$ and any $\phi \in C_0^\infty(\mathbb{R}^3)$:*

$$\int \frac{|\phi|^2}{(1+|t-r|)^2} \frac{w}{(1+t+r)^{1-a}} dx \lesssim \int |\partial \phi|^2 \frac{w}{(1+t+r)^{1-a}} dx$$

Finally we will also use this lemma, the notations L_μ and $\overline{\partial}$ appearing in the lemma will be explained in the subsection entitled "The null frame".

Lemma 9.7. *Let $\partial_q = \frac{\partial_r - \partial_t}{2}$ and let $\overline{\partial}_\mu = \partial_\mu - L_\mu \partial_q$ be the projection of ∂_μ onto the tangent space of the outgoing light cones. Then*

$$|\partial \phi(t,x)| \lesssim |\partial_q \phi(t,x)| + |\overline{\partial} \phi(t,x)| \tag{9.6}$$

$$(1+|t-r|)|\partial \phi(t,x)| + (1+t+r)|\overline{\partial}\phi(t,x)| \lesssim |\sum_{|I|=1} |Z^I \phi(t,x)| \tag{9.7}$$

Proposition 9.2. *Suppose that $N \geq 4$ and the weak energy bounds (9.1) hold for a solution of Einstein-Vlasov in wave coordinates. Then :*

$$|Z^I h^1(t,x)| \leq \frac{C\epsilon(1+t)^{2\delta}}{(1+t+r)(1+q_+)^\gamma}, \qquad |I| \leq N-3. \tag{9.8}$$

where $q = r - t$, $q_+ = max\{q, 0\}$, $q_- = max\{-q, 0\}$. Moreover

$$(1+|q|)|\partial Z^I h^1(t,x)| + (1+t+r)|\overline{\partial} Z^I h^1(t,x)| \leq \frac{C\epsilon(1+t)^{2\delta}}{(1+t+r)(1+q_+)^\gamma} \qquad |I| \leq N-4. \tag{9.9}$$

The same estimates hold for H_1 in place of h^1, and for h or H in place of h_1 if γ is replaced by 2δ.

Proof. The bound (9.9) comes from (9.8) using (9.7). It only remains to prove (9.8). We consider the case $r < t$ (the case $r \geq t$ is then straightforward). First note that :

$$\Box h_{\mu\nu} = \tilde{\Box}_g h_{\mu\nu} - \tilde{\Box}_g h_{\mu\nu} + \Box h_{\mu\nu} = \tilde{\Box}_g h_{\mu\nu} - (g^{\alpha\beta} - m^{\alpha\beta})\partial_\alpha\partial_\beta h^{\mu\nu} = \tilde{\Box}_g h_{\mu\nu} - H^{\alpha\beta}\partial_\alpha\partial_\beta h^{\mu\nu}$$

Then

$$\Box h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h) + \tilde{T}_{\mu\nu} - H^{\alpha\beta}\partial_\alpha\partial_\beta h^{\mu\nu}$$

Hence

$$\Box h^1_{\mu\nu} = -H^{\alpha\beta}\partial_\alpha\partial_\beta h^{\mu\nu} + F_{\mu\nu}(h)(\partial h, \partial h) - \Box h^0_{\mu\nu} + \tilde{T}_{\mu\nu}$$

We now write $h^1_{\mu\nu} = v_{\mu\nu} + u_{\mu\nu} + \phi_{\mu\nu}$ where,

$$\Box v_{\mu\nu} = 0 \qquad v_{\mu\nu}|_{t=0} = h^1_{\mu\nu}|_{t=0}, \qquad \partial_t v_{\mu\nu}|_{t=0} = \partial_t h^1_{\mu\nu}|_{t=0}$$

and

$$\Box u_{\mu\nu} = -H^{\alpha\beta}\partial_\alpha\partial_\beta h^{\mu\nu} + F_{\mu\nu}(h)(\partial h, \partial h) - \Box h^0_{\mu\nu} \qquad u_{\mu\nu}|_{t=0} = \partial_t u_{\mu\nu}|_{t=0} = 0$$

$$\Box \phi_{\mu\nu} = \tilde{T}_{\mu\nu} \qquad \phi_{\mu\nu}|_{t=0} = \partial_t \phi_{\mu\nu}|_{t=0} = 0$$

This decomposition is used so that we can apply lemma 9.4 for v and lemma 9.5 for u and ϕ. We will prove (9.8) for each v, u and ϕ. For v with lemma 9.5 we have

$$(1+t)|v(t,x)| \leq C\sup_x((1+|x|)^{2+\gamma}(|v_1(x)| + |\partial v_0(x)|) + (1+|x|)^{1+\gamma}|v_0(x)|)$$

and with proposition 9.1 we have estimates on v_0 and v_1 for $|I| = 0$

Hence for $b > 0$. Since we take the estimate at $t = 0$ from (9.4) and (9.5) we must consider the case $r > t$ in those equations. Hence

$$\begin{aligned}(1+t)|v(t,x)| &\leq \frac{C\epsilon(1+r)^{2+b}}{(1+r)^{2+\gamma}}.\\ (1+t)|v(t,x)| &\leq C\epsilon(1+r)^{b-\gamma}\\ |v(t,x)| &\leq \frac{C\epsilon(1+r)^{b-\gamma}}{(1+t)}\end{aligned}$$

And with $b < 2\delta$

$$|v(t,x)| \leq \frac{C\epsilon(1+r)^{2\delta}}{(1+t)(1+r)^{\gamma}}$$

With $r < t$

$$|v(t,x)| \leq \frac{C\epsilon(1+t)^{2\delta}}{(1+t+r)(1+q_+)^{\gamma}}$$

Now, since our vector fields are Killing $Z^I v$ satisfy the same equation than v and with initial condition with $Z^I h^1$. Using the same proof as above and proposition (9.1) with $|I| \leq N-2$ we get the result for v. It remains to check the bounds for u and ϕ.

From Lemma 9.4 we have

$$|u(t,x)|(1+t+|x|) \leq C\sum_{|I|\leq 2}\int_0^t\int_{\mathbb{R}^3}\frac{|Z^I K(s,y)|}{1+s+|y|}dyds,$$

where $K = -H^{\alpha\beta}\partial_\alpha\partial_\beta h^{\mu\nu} + F_{\mu\nu}(h)(\partial h, \partial h) - \Box h^0_{\mu\nu}$

Recalling that $F_{\mu\nu}(u)(v,v)$ depends quadratically on v and the bounds from proposition (9.1) we have the following rough inequality

$$|Z^I F_{\mu\nu}(\partial h, \partial h)| \leq C\sum_{|J|+|K|\leq|I|}|\partial Z^J h||\partial Z^K h| + C\sum_{|J|+|K|\leq|I|}\frac{|\partial Z^J h|}{1+|q|}|\partial Z^K h|$$

Moreover since $H^{\alpha\beta} = -h_{\alpha\beta} + O(h^2)$

we have

$$|Z^I(H^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu})| \leq C\sum_{|J|+|K|\leq|I|+1,\ |J|\leq|I|}\frac{|\partial Z^J h|}{1+|q|}|\partial Z^K h|$$

Now recall

$$|\partial Z^I h^1(t,x)| \le \begin{cases} \frac{C\epsilon(1+t)^\delta}{(1+t+r)(1+|r-t|)^{1+\gamma}}, & r>t \\ \\ \frac{C\epsilon(1+t)^\delta}{(1+t+r)(1+|r-t|)^{1/2}}, & r<t \end{cases}, \qquad |I| \le N-2$$

and that the same estimates hold for h instead of h^1 where γ is replaced by δ.
Now $x \mapsto \partial Z^I h(t,x)$ is in L^2 and with Cauchy-Schwarz inequality we get

$$\int |\partial Z^J h||\partial Z^K h|(s,y)dy \le \sum_{|I|\le N} \|\partial Z^I h(s,.)\|_{L^2}^2 \le C\epsilon^2(1+s)^{2\delta}$$

Since we have

$$\int \frac{|Z^J h^0(t,x)|^2}{(1+|t-r|)^2}dx \le CM^2 \int_0^\infty \frac{r^2 dr}{(1+|t+r|)^2(1+|t-r|)^2} \le C'M^2$$

where the spherical coordinates have been used.
and with the Hardy's inequality (lemma 9.6) :

$$\int \frac{|Z^J h^1(t,x)|^2}{(1+|t-r|)^2}dx \le C\int |\partial Z^J h^1(t,x)|^2 dx \le C\epsilon^2(1+t)^{2\delta}$$

Hence with Cauchy-Schwarz inequality and the previous results

$$\int \frac{|Z^J h|}{(1+q)}|\partial Z^K h|(s,y)dy \le C\epsilon^2(1+t)^{2\delta}$$

Recall the definition of $h^0_{\mu\nu}$

$$h^0_{\mu\nu}(t,x) = \chi(\frac{r}{t})\chi(r)\frac{M}{r}\delta_{\mu\nu}$$

$\chi(r)\chi(r/t)$ implies that the derivative are not 0 when $1/2<r<3/4$ and $t/2<r<3t/4$. Hence we can see it as a smooth cut off function $\chi(\frac{r}{1+t})$. Since $\Box\frac{M}{r}=0$ away from the zone $r=0$ we have

$$\Box h^0_{\mu\nu} = \left|\frac{M\delta_{\mu\nu}}{r}(\partial t+\partial r)(\partial t-\partial r)(\chi(\frac{r}{1+t}))\right|$$

The second order partial derivative with respect to t gives term in $\frac{r}{(1+t)^3}$ and $\frac{r^2}{(1+t)^4}$. The The second order partial derivative with respect to r gives terms in $\frac{1}{(1+t)^2}$. Applying $\frac{M}{r}$ to those terms we get

$$\Box h^0_{\mu\nu} \le \frac{CMH(r<3t/4)}{(1+t+r)^3}$$

where $H(r<3t/4)=1$ when $r<3t/4$ and 0 otherwise. Hence

$$\|\Box h^0(t,.)\|_{L^1} \le M$$

From Hörmander inequality on u we get

$$|u_{\mu\nu}(t,x)|(1+t+|x|) \le C\epsilon(1+t)^{2\delta}$$

From the same inequality and the bound (9.1) we have

$$|u_{\mu\nu}(t,x)|(1+t+|x|) \le C\epsilon(1+t)^{\delta}$$

Proposition 9.2 follows from applying the same ideas than for v considering $Z^I u$ and $Z^I\phi$.

$\square$

Lemma 9.8. *If supp $\phi \subset \{(t,x); |x| \le K+ct\}$ for some $K\ge 0$ and $0<c<1$ then*

$$|\phi(t,x)| \lesssim (1+t+r)^{-3}\sum_{|I|=3}\|Z^I\phi(t,.)\|_{L^1}.$$

Proof. For $t \le \frac{2K}{1-c}$, from Sobolev inequality :

$$\begin{aligned}
|\Phi(t,x)| &\lesssim \sum_{|I|=3} |\partial^I \phi(t,.)| \\
&\lesssim \frac{1}{(1+\frac{2K}{1-c}+K+c\frac{2K}{1-c})^3} \sum_{|I|=3} |\partial^I \phi(t,.)| \\
&\lesssim (1+t+r)^{-3} \sum_{|I|=3} \|Z^I \phi(t,.)\|_{L^1}
\end{aligned}$$

Now for $t \ge \frac{2K}{1-c}$ we have supp $\phi \subset \{(t,x); |x|c't\}$ where $c' = \frac{1+c}{2}$. From identity (8.12) we get for $|I| = 3$

$$|\partial^I| \lesssim \frac{1}{t^3} \sum |Z^J|$$

Moreover since

$$\begin{aligned}
1 + |x|/t + 1/t &\le 1 + c' + (1-c)/K \\
t + |x| + 1 &\le ct \\
\frac{1}{t} &\lesssim \frac{1}{t+|x|+1}
\end{aligned}$$

and lemma 9.8 follows from the Sobolev inequality.

□

The previous Lemma and the assumption supp$\hat{T} \subset \{(t,x); |x| \le K + ct\}$, $c < 1$ give

(9.10) $$|Z^I T(t,x)| \lesssim \epsilon(1+t)^3 \qquad |I| \le N-4$$

Before beginning the subsection on the sharp decay estimates for the first order derivatives we need to understand the null frame.

9.3 The null frame

Whether a given quasilinear wave equation admits global solution for small data or not depends on the structure on the nonlinear terms. Here $\tilde{T}^{\mu\nu}$ vanishes at late times and so plays no role un this discussion. The tensor $h_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu}$ verify

$$\widetilde{\Box}_g h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h) + \widehat{T}_{\mu\nu}$$

And we can verify (see the vacuum case in section 3) that

(9.11) $$F_{\mu\nu}(h)(\partial h, \partial h) = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h)$$

where

$$\begin{aligned}
P(\partial_\mu h, \partial_\nu h) &= \frac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \partial_\mu h_{\alpha\beta} \partial_\nu h_{\alpha'\beta'} - \frac{1}{4} m^{\alpha\alpha'} \partial_\mu h_{\alpha\alpha'} m^{\beta\beta'} \partial_\nu h_{\beta\beta'} \\
P(\partial_\mu h, \partial_\nu h) &= \frac{1}{2} \partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta} - \frac{1}{4} \partial_\mu trh \partial_\nu trh
\end{aligned}$$

Here we have $|Q_{\mu\nu}(\partial h, \partial h)| \lesssim |\overline{\partial} h||\partial h|$ and $|G_{\mu\nu}(h)(\partial h, \partial h)| \lesssim |h||\partial h|^2$
The weak decay estimates of the previous subsection imply that

$$|Q_{\mu\nu}(\partial h, \partial h)| + |G_{\mu\nu}(h)(\partial h, \partial h)| \lesssim |\overline{\partial} h||\partial h| + |h||\partial h|^2 \lesssim \frac{\epsilon^2}{(1+t+r)^{2+2\gamma}(1+|t-r|)^{2\gamma}}$$

The problem arises in the $P(\partial_\mu h, \partial_\nu h)$ terms. Indeed the weak decay estimates only provide a decay of

$$|P(\partial_\mu h, \partial_\nu h)| \lesssim \frac{\epsilon^2}{(1+t+r)^{1+2\gamma}(1+|t-r|)^{1+2\gamma}}$$

which is not sufficient in the wave zone (i.e $t \sim r$). To improve this result we need to decompose the semilinear terms in a null frame we shall introduce and work on its components. It is in fact well known that for solutions of the wave equation, derivatives tangential to the outgoing light cones $\overline{\partial}$ decay faster.

At each point (t, x) consider the pair of null vectors $(L, \underline{L})$ such that,

$$L^0 = 1, \quad L^i = x^i/|x|, \quad i = 1,2,3, \quad \underline{L}^0 = 1 \quad \underline{L}^i = -x^i/|x|, \quad i = 1,2,3$$

We then add two vectors S_1 and S_2 of the sphere S^2 orthogonal to $\omega = x/|x|$. Hence we have a null frame $\mathcal{N} = \{\underline{L}, L, S_1, S_2\}$. We denote $\overline{\partial} \in \{\partial_L, \partial_{S_1}, \partial_{S_2}\}$ In what follows we will denote by A, B any of the vectorfields S_1, S_2.

We use the summation conventions :

$$X^A A^\alpha = X^\beta S_{1\beta} S_1^\alpha + X^\beta S_{2\beta} S_2^\alpha \qquad X_A Y_A = X^\alpha Y^\beta S_{1\alpha} S_{1\beta} + X^\alpha Y^\beta S_{2\alpha} S_{2\beta}$$

If X is a vector field : $X = X^\alpha \partial_\alpha$, then its decomposition it can be decompose in the null frame with :

$$X^\alpha = X^L L^\alpha + X^{\underline{L}} \underline{L}^\alpha + X^A A^\alpha$$

For instance, given two vector fields X and Y we have :

$$m_{\alpha\beta} X^\alpha Y^\beta = -(X^L + X^{\underline{L}} + X^A A^0)(Y^L + Y^{\underline{L}} + Y^A A^0) + \sum_{i=1}^{3} \left(X^L \frac{x^i}{|x|} - X^{\underline{L}} \frac{x^i}{|x|} + X^A A^i \right) \left(Y^L \frac{x^i}{|x|} - Y^{\underline{L}} \frac{x^i}{|x|} + Y^A A^i \right)$$

and

$$\sum_{i=1}^{3} \left(X^L \frac{x^i}{|x|} - X^{\underline{L}} \frac{x^i}{|x|} + X^A A^i \right) \left(Y^L \frac{x^i}{|x|} - Y^{\underline{L}} \frac{x^i}{|x|} + Y^A A^i \right)$$
$$= X^L Y^L + X^{\underline{L}} Y^{\underline{L}} - X^L Y^{\underline{L}} - X^{\underline{L}} Y^L + \sum_{i=1}^{3} \left(X^L \frac{x^i}{|x|} Y^A A^i - X^{\underline{L}} \frac{x^i}{|x|} Y^A A^i + X^A A^i Y^L \frac{x^i}{|x|} - X^A A^i Y^L \frac{x^i}{|x|} + X^A A^i Y^A A^i \right)$$

Finally we have

$$m_{\alpha\beta} X^\alpha Y^\beta = -2(X^L Y^{\underline{L}} + X^{\underline{L}} Y^L) + X^A Y^A$$

Hence, relative to the null frame the Minkowski metric has the following form :

$$m_{LL} = m_{\underline{L}\underline{L}} = m_{LA} = m_{\underline{L}A} = 0, \quad m_{L\underline{L}} = m_{\underline{L}L} = -2, \quad m_{AB} = \delta_{AB}$$

We have the following notations for vector fields : $X_\alpha = m_{\alpha\beta} X^\beta$ and we define $X_Y = m_{\alpha\beta} X^\alpha Y^\beta = X_\alpha Y^\alpha$. We have

$$X_\alpha = X^L L_\alpha + X^{\underline{L}} \underline{L}_\alpha + X^A A_\alpha$$

And we can show that

$$m_{\alpha\beta} X_\alpha Y_\beta = -\frac{1}{2}(X_L Y_{\underline{L}} + X_{\underline{L}} Y_L) + X_A Y_A$$

hence

$$m^{LL} = m^{\underline{L}\underline{L}} = m^{LA} = m^{\underline{L}A} = 0, \quad m^{L\underline{L}} = m^{\underline{L}L} = -\frac{1}{2}, \quad m^{AB} = \delta^{AB}$$

In fact since

$$L^\alpha \underline{L}_\alpha = -2 \quad L^\alpha L_\alpha = \underline{L}^\alpha \underline{L}_\alpha = 0, \quad L^\alpha A_\alpha = \underline{L}^\alpha A_\alpha = 0, and A^\alpha B_\alpha = \delta_{AB}$$

we have

$$X^L = -\frac{1}{2} X_{\underline{L}}, \quad X^{\underline{L}} = -\frac{1}{2} X_L, \quad X^A = X_A$$

We define $k_{UV} = U^\mu V^\nu k_{\mu\nu}$ and $k^{UV} = U_\mu V_\nu k^{\mu\nu}$ where U and V are two vectors and the previous results on the Minkowski's metric follow.

Finally we define

$$\not{tr} D = \delta^{AB} D_{AB}$$

9.4 The sharp decay estimates for the first order derivatives

In this subsection we assume that proposition 9.2 holds we we don't need (9.1) to hold any further. Recall the wave coordinate condition which leads to

$$g^{\alpha\beta}\partial_\alpha g_{\beta\mu} = \frac{1}{2}g^{\alpha\beta}\partial_\mu g_{\alpha\beta}$$

This condition can also be written as :

$$\partial_\mu\big(g^{\mu\nu}\sqrt{|detg|}\big) = 0.$$

We have

$$|detg| = |det(m+H)| = |det(I+Hm^{-1})| = 1 - tr_m H + O(H^2)$$

Hence

$$g^{\mu\nu}\sqrt{|detg|} = (m^{\mu\nu}+H^{\mu\nu})(1-\frac{1}{2}tr_m H + O(H^2))$$

And the wave coordinate equation becomes :

$$\partial_\mu\Big(H^{\mu\nu} - \frac{1}{2}m^{\mu\nu}trH + O^{\mu\nu}(H^2)\Big) = 0.$$

which can be written :

$$\partial_\mu \hat{H}^{\mu\nu} = W^\nu(h,\partial h) \qquad \text{where} \qquad \hat{H}^{\mu\nu} = H^{\mu\nu} - m^{\mu\nu}tr_m H/2$$

and $|W(h,\partial h)| \lesssim |h||\partial h|$.
If we recall the decomposition

$$g^{\mu\nu} = m^{\mu\nu} + H^{\mu\nu} \qquad H^{\mu\nu} = H_0^{\mu\nu} + H_1^{\mu\nu} \qquad H_0^{\mu\nu} = -\chi(\frac{r}{1+t})\frac{M}{r}\delta^{\mu\nu}$$

Then we have

$$\partial_\mu \hat{H}_1^{\mu\nu} = W^\nu(h,\partial h) - \partial_\mu \hat{H}_0^{\mu\nu} \qquad \text{where} \qquad \partial_\mu \hat{H}_0^{\mu\nu} = 2\chi'(\frac{r}{1+t})M(1+t)^{-2}\delta^{0\nu}$$

Lemma 9.9. *Let $\partial_q = (\partial_r - \partial_t)/2$ and $\partial_s = (\partial_r + \partial_t)/2$. Then for any tensor $k^{\mu\nu}$:*

$$\partial_q(L_\mu U_\nu k^{\mu\nu}) = L_\mu U_\nu \partial_q k^{\mu\nu} = \underline{L}_\mu U_\nu \partial_s k^{\mu\nu} - A_\mu U_\nu \partial_A k^{\mu\nu} + U_\nu \partial_\mu k^{\mu\nu} \qquad U \in \mathcal{N}$$

Proof. We can notice that $\partial_q = -\frac{1}{2}\underline{L}^\alpha\partial_\alpha$ and $\partial_s = \frac{1}{2}L^\alpha\partial_\alpha$. From the previous subsection it is clear that the divergence of F can be expressed in the null frame with

$$\partial_\mu F^\mu = L_\mu \partial_q F^\mu - \underline{L}_\mu \partial_s F^\mu + A_\mu \partial_A F^\mu$$

Since ∂_s and ∂_q commute with the frame we let $F = U_v k^{\mu\nu}$ and substrace in the previous inequality. The result follows.

□

Lemma 9.10. *We have*

$$|\partial_q H_{LT}| + |\partial_q \cancel{tr} H| \lesssim |\overline{\partial} H| + |h||\partial h|, \tag{9.12}$$

$$|\partial_q H_{1LT}| + |\partial_q \cancel{tr} H_1| \lesssim |\overline{\partial} H_1| + |h||\partial h| + M\big|\chi'(\frac{r}{t+1})\big|(1+t+r)^{-2}, \tag{9.13}$$

where $\chi'(s)$, is a function supported when $1/4 \leq s \leq 1/2$ and $T = \{L, S_1, S_2\}$. Moreover, (9.12) hold also for h in place of H.

Proof. From the previous lemma we get

$$\partial_q \hat{H}_{1\,LU}^{\mu\nu} = \underline{L}_\mu U_\nu \partial_s \hat{H}_1^{\mu\nu} - A_\mu U_\nu \partial_A \hat{H}_1^{\mu\nu} + U_\nu \partial_\mu \hat{H}_1^{\mu\nu}$$

Hence

$$|\partial_q \hat{H}_{1\,LU}| \lesssim |\overline{\partial} H_1| + |\partial_\mu \hat{H}_1|$$

From

$$\partial_\mu \hat{H}_1^{\mu\nu} = W^\nu(h, \partial h) - \partial_\mu \hat{H}_0^{\mu\nu} \qquad \text{where} \qquad \partial_\mu \hat{H}_0^{\mu\nu} = 2\chi'(\frac{r}{1+t})M(1+t)^{-2}\delta^{0\nu}$$

and $|W(h, \partial h)| \lesssim |h||\partial h|$ we get

$$|\partial_q \hat{H}_{1\,LU}| \lesssim |\overline{\partial} H_1| + |h||\partial h| + M\big|\chi'(\frac{r}{t+1})\big|(1+t+r)^{-2}$$

Recalling that

$$\hat{H}^{\mu\nu} = H^{\mu\nu} - m^{\mu\nu} tr_m H/2$$

and picking $U = T$ and $U = \underline{L}$ gives (6.17). (6.16) is easily obtained in the same way.

□

Proposition 9.3. *We have*

(9.14)
$$|\partial_q H_{1LT}| + |\partial_q \cancel{tr} H_1| \lesssim \epsilon(1+t+r)^{-2+2\delta}(1+q_+)^{-\gamma},$$

(9.15) $|H_{1LT}| + |\cancel{tr} H_1| \lesssim \epsilon(1+t+r)^{-1-\gamma+2\delta} + \epsilon(1+t)^{-2+2\delta}(1+q_-) \lesssim \epsilon(1+t+r)^{-1-\gamma+2\delta}(1+q_-)^\gamma$

The same estimates hold for H in place of H_1 if γ is replaced by 2δ.

Proof. We first show (9.14). From Proposition 9.2

$$(1+|q|)|\partial Z^I H_1(t,x)| + (1+t+r)|\overline{\partial} Z^I H_1(t,x)| \leq \frac{C\epsilon(1+t)^{2\delta}}{(1+t+r)(1+q_+)^\gamma} \qquad |I| \leq N-4.$$

For $|t-r| > t/8$ we have

$$|\partial_q H_1(t,x)| \leq \frac{C\epsilon(1+t+r)^{2\delta}}{(1+t+r)(1+q_+)^\gamma(1+|t-r|)}$$

Suppose $r > t$:
Then

$$\frac{1+t+r}{1+r-t} \leq \frac{1+3r}{1+r} \leq C$$

Suppose $t > r$

$$\frac{1+t+r}{1+|t-r|} \leq \frac{1+2t}{1+t/8} \leq C$$

Hence

$$|\partial_q H_1(t,x)| \lesssim \epsilon(1+t+r)^{-2+2\delta}(1+q_+)^{-\gamma}$$

In a similar way :

$$|\partial_q \cancel{tr} H_1(t,x)| \lesssim \epsilon(1+t+r)^{-2+2\delta}(1+q_+)^{-\gamma}$$

For $t - r < t/8$ we use the previous Lemma along with proposition 9.2 :

$$|\partial_q H_{1LT}| + |\partial_q \cancel{tr} H_1| \lesssim |\overline{\partial} H_1| + |h||\partial h| \lesssim \frac{\epsilon}{(1+t+r)^{2-2\delta}(1+q_+)^\gamma} + \frac{\epsilon^2}{(1+t+r)^{2-2\delta}(1+q_+)^{2\gamma+1}}$$

The proposition follows.

□

Recall that

$$F_{\mu\nu}(h)(\partial h, \partial h) = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h)$$

where

$$P(h,k) = \frac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} h_{\alpha\beta} k_{\alpha'\beta'} - \frac{1}{4} m^{\alpha\alpha'} h_{\alpha\alpha'} m^{\beta\beta'} k_{\beta\beta'}$$

and $|Q_{\mu\nu}(\partial h, \partial k)| \lesssim |\overline{\partial} h||\partial k| + |\partial h||\overline{\partial} k|$, $|G_{\mu\nu}(h)(\partial h, \partial h)| \lesssim |h||\partial h|^2$
We can note that

(9.16) $$|P(\partial_\mu h, \partial_\nu k) - L_\mu L_\nu P(\partial_q h, \partial_q h)| \lesssim |\overline{\partial} h||\partial k| + |\partial h||\overline{\partial} k|$$

Now we denote $P_{\mathcal{N}}(h,k) := P(h,k)$ the decomposition of P in a null frame and we have

Proposition 9.4.

$$P_{\mathcal{N}}(h,k) = -\frac{1}{8}(h_{LL}k_{\underline{L}\underline{L}} + h_{\underline{L}\underline{L}}k_{LL}) - \frac{1}{4}\delta^{CD}\delta^{C'D'}(2h_{CC'}k_{DD'} - h_{CD}k_{C'D'}) + \frac{1}{4}\delta^{CD}(2h_{CL}k_{D\underline{L}} + 2h_{C\underline{L}}k_{DL} - h_{CD}k_{L\underline{L}} - h_{L\underline{L}}k_{CD}) \tag{9.17}$$

Proof. Recall from the subsection on the null frame that we have shown that

$$m^{LL} = m^{\underline{L}\underline{L}} = m^{LA} = m^{\underline{L}A} = 0, \quad m^{L\underline{L}} = m^{\underline{L}L} = -\frac{1}{2}, \quad m^{AB} = \delta^{AB}$$

Hence, if we let $k_{XY} = k_{\alpha\beta}X^\alpha Y^\beta$ then

$$tr_m k = m^{\alpha\beta}k_{\alpha\beta} = -\frac{1}{2}(k_{L\underline{L}} + k_{\underline{L}L}) + \cancel{tr}k$$

where

$$\cancel{tr}k = \delta^{AB}k_{AB}$$

And if we recall the definition of S_1 and S_2 it follows

$$\cancel{tr}k = (\delta^{ij} - \omega^i\omega^j)k_{ij}$$

for $i,j = 1,2,3$ Hence if k and p are symmetric :

$$\begin{aligned} p_{\alpha\beta}k^{\alpha\beta} &= m^{\alpha\alpha'}m^{\beta\beta'}p_{\alpha\beta}k_{\alpha'\beta'} \\ &= \frac{1}{4}(p_{LL}k_{\underline{L}\underline{L}} + p_{\underline{L}\underline{L}}k_{LL} + 2p_{L\underline{L}}k_{\underline{L}L}) - \delta^{AB}(p_{AL}k_{B\underline{L}} + p_{A\underline{L}}k_{BL}) + \delta^{AB}\delta^{A'B'}p_{AA'}k_{BB'} \end{aligned}$$

Hence from

$$P_{\mathcal{N}}(p,k) = \frac{1}{2}m^{\alpha\alpha'}m^{\beta\beta'}h_{\alpha\beta}k_{\alpha'\beta'} - \frac{1}{4}m^{\alpha\alpha'}h_{\alpha\alpha'}m^{\beta\beta'}k_{\beta\beta'}$$

we get

$$P_{\mathcal{N}}(h,k) = -\frac{1}{8}(h_{LL}k_{\underline{L}\underline{L}} + h_{\underline{L}\underline{L}}k_{LL}) - \frac{1}{4}\delta^{CD}\delta^{C'D'}(2h_{CC'}k_{DD'} - h_{CD}k_{C'D'}) + \frac{1}{4}\delta^{CD}(2h_{CL}k_{D\underline{L}} + 2h_{C\underline{L}}k_{DL} - h_{CD}k_{L\underline{L}} - h_{L\underline{L}}k_{CD})$$

which ends the proof.

□

Now with

$$P_{\mathcal{S}}(D,E) = -\hat{D}_{AB}\hat{E}^{AB}/2, \quad A,B \in \mathcal{S}, \quad \hat{D}_{AB} = D_{AB} - \delta_{AB}\cancel{tr}D/2$$

we have

$$|P_{\mathcal{N}}(h,k) - P_{\mathcal{S}}(h,k)| \lesssim (|h|_{LT} + |\cancel{tr}h|)|k| + |h|(|k|_{LT} + |\cancel{tr}k|), \quad \text{where} \quad |h|_{LT} = |h_{LL}| + |h_{LS_1}| + |h_{LS_2}| \tag{9.18}$$

and from (9.12) with $H = -h + O(h^2)$ we get

$$|P_{\mathcal{N}}(h,k) - P_{\mathcal{S}}(h,k)| \lesssim (|\overline{\partial}h| + |h||\partial h|)|\partial h|$$

Hence if we write

$$\cancel{P}_{\mu\nu}(\partial h, \partial h) = \overline{\chi}\Big(\frac{\langle r-t\rangle}{t+r}\Big)L_\mu L_\nu P_{\mathcal{S}}(\partial_q h, \partial_q h) \tag{9.19}$$

where $\overline{\chi} \in C_0^\infty$ satisfies $\overline{\chi}(q) = 0$, when $|q| \geq 3/4$ and $\overline{\chi}(q) = 1$, when $|q| \leq 1/2$ we have

Lemma 9.11.

$$|F_{\mu\nu}(h)(\partial h, \partial h) - \cancel{P}_{\mu\nu}(\partial h, \partial h)| \lesssim |\overline{\partial}h||\partial h| + |h||\partial h|^2, \qquad \textit{when} \quad |q| \leq 1/2.$$

Proof. We have

$$F_{\mu\nu}(h)(\partial h,\partial h) = P(\partial_\mu h,\partial_\nu h) + Q_{\mu\nu}(\partial h,\partial h) + G_{\mu\nu}(h)(\partial h,\partial h)$$

where $|Q_{\mu\nu}(\partial h,\partial k)| \lesssim |\overline{\partial} h||\partial k| + |\partial h||\overline{\partial} k|$, $|G_{\mu\nu}(h)(\partial h,\partial h)| \lesssim |h||\partial h|^2$.

With the bounds (9.16) and (9.18) the result follows.

□

Now with the bounds (9.4), (9.5), (9.8) and (9.9) we have

Lemma 9.12.

$$|F_{\mu\nu}(h)(\partial h,\partial h) - \not{P}_{\mu\nu}(\partial h,\partial h)| \lesssim \frac{\epsilon^2}{(1+t+r)^{3-4\delta}(1+|q|)(1+q_+)^{4\delta}}$$

Lemma 9.13. *If $k^{\alpha\beta}$ is a symmetric tensor and ϕ a function then*

$$|k^{\alpha\beta}\partial_\alpha\partial_\beta\phi| \lesssim \Big(\frac{|k_{LL}|}{1+|q|} + \frac{|k|}{1+t+r}\Big)\sum_{|K|\le 1}|\partial Z^K\phi|$$

Proof. Rrcall that $\overline{\partial} \in \{L, S_1, S_2\}$
and that

$$p_{\alpha\beta}k^{\alpha\beta} = \frac{1}{4}(p_{LL}k_{\underline{L}\underline{L}} + p_{\underline{L}\underline{L}}k_{LL} + 2p_{L\underline{L}}k_{\underline{L}L}) - \delta^{AB}(p_{AL}k_{B\underline{L}} + p_{A\underline{L}}k_{BL}) + \delta^{AB}\delta^{A'B'}p_{AA'}k_{BB'}$$

□

Applying that for our expression in the Lemma we get :

$$|k^{\alpha\beta}\partial_\alpha\partial_\beta\phi| \lesssim (|k|_{LL}|\partial^2\phi| + |k||\overline{\partial}\partial\phi|)$$

Now from (9.7) applied to $\partial\phi$ instead of ϕ we get the result.

Lemma 9.14. *We have*

$$|(\tilde{\Box}_g - \Box_0)\phi| \lesssim \frac{\epsilon(1+q_+)^{-\gamma}}{(1+t+r)^{1+\gamma-2\delta}(1+|q|)^{1-\gamma}}\sum_{K\le 1}|\partial Z^K\phi|$$

where the asymptotic Schwarzschild wave operator is given by

$$\Box_0 = (m^{\alpha\beta} + H_0^{\alpha\beta})\partial_\alpha\partial_\beta, \quad \textit{where} \quad H_0^{\alpha\beta} = -\chi\big(\frac{r}{1+t}\big)\frac{M}{r}\delta^{\alpha\beta}$$

Hence, instead of using the the full wave operator $\tilde{\Box}_g$ we will use the estimate for the operator $\Box_0$.

Proof. We have

$$|(\tilde{\Box}_g - \Box_0)\phi| = |H_1^{\alpha\beta}\partial_\alpha\partial_\beta\phi|$$

Now from the previous lemma we get

$$|H_1^{\alpha\beta}\partial_\alpha\partial_\beta\phi| \lesssim \Big(\frac{|H_{1LL}|}{1+|q|} + \frac{|H_1|}{1+t+r}\Big)\sum_{|K|\le 1}|\partial Z^K\phi|$$

From (9.15) and (9.8) :

$$|H_{1LL}| \lesssim \epsilon(1+t+r)^{-1-\gamma+2\delta}(1+q_-)^\gamma$$

$$|H_1| \le \frac{C\epsilon(1+t)^{2\delta}}{(1+t+r)(1+q_+)^\gamma}$$

Hence

$$\begin{aligned}
|H_1^{\alpha\beta}\partial_\alpha\partial_\beta\phi| &\lesssim \frac{\epsilon(1+t+r)^2(1+q_+)^\gamma + C\epsilon(1+t)^{2\delta}}{(1+|q|)(1+q^+)^{2\gamma}(1+t+r)^{\gamma+1}}\sum_{|K|\le 1}|\partial Z^K\phi| \\
&\lesssim \frac{C\epsilon(1+q_+)^\gamma(1+t+r)^{2\delta}}{(1+|q|)(1+q^+)^{2\gamma}(1+t+r)^{\gamma+1}}\sum_{|K|\le 1}|\partial Z^K\phi| \\
&\lesssim \frac{C\epsilon}{(1+t+r)^{1+\gamma-2\delta}(1+|q|)^{1-\gamma}(1+q_+)^\gamma}\sum_{K\le 1}|\partial Z^K\phi|
\end{aligned}$$

The lemma follows.

□

In spherical coordinates this takes the form

(9.20)
$$\Box_0\phi = \Big(-\partial_t^2+\Delta_x-\frac{M}{r}\chi\Big(\frac{r}{1+t}\Big)(\partial_t^2+\Delta_x)\Big)\phi = \frac{1}{r}\Big(-\partial_t^2+\partial_r^2-\frac{M}{r}\chi\Big(\frac{r}{1+t}\Big)(\partial_t^2+\partial_r^2)\Big)(r\phi)+\Big(1-\frac{M}{r}\chi\Big(\frac{r}{1+t}\Big)\Big)\frac{1}{r^2}\Delta_\omega\phi$$

where $\Delta_\omega = \sum_{i,j}\Omega_{ij}^2$. With

$$h^0_{\mu\nu} = \frac{M}{r}\chi\Big(\frac{r}{1+t}\Big)\delta_{\mu\nu}$$

we get

$$\Box_0 h^0_{\mu\nu} = \frac{M\delta_{\mu\nu}}{(1+t)^3}\Big(\chi_1'\Big(\frac{r}{1+t}\Big)+\frac{1}{t+1}\chi_2'\Big(\frac{r}{1+t}\Big)\Big) \tag{9.21}$$

for some functions $\chi_i'(s)$ that vanish when $s \geq 1/2$ or $s \leq 1/4$. Hence in the wave zone $t \sim r$ and in the exterior region, h^0 is a solution of the asymptotic Schwarzschild operator. Moreover with the definition of $h^0_{\mu\nu}$, there exists smooth cut-off functions $\chi_{\alpha\beta}(s,\omega)$ supported when $s \geq 1/4$ such that

$$|(\tilde{\Box}_g-\Box_0)h^0_{\mu\nu}| = \frac{M\delta_{\mu\nu}}{(1+t+r)^3}H_1^{\alpha\beta}\chi_{\alpha\beta}\Big(\frac{r}{t+1},w\Big) \tag{9.22}$$

Hence with the bound (9.8) we get

$$|(\tilde{\Box}_g-\Box_0)h^0_{\mu\nu}| \lesssim \frac{\epsilon M}{(1+t+r)^{4-2\delta}(1+q_+)^\gamma} \tag{9.23}$$

Proposition 9.5. *When $|x| \geq ct$ we have*

$$|\Box_0 h^1_{\mu\nu} - \not{P}_{\mu\nu}(\partial h,\partial h)| \lesssim \frac{\epsilon^2}{(1+t+r)^{2+\gamma-4\delta}(1+|q|)^{2-\gamma}(1+q_+)^{4\delta}}$$

Proof. We have

$$\begin{aligned}\Box_0 h^1_{\mu\nu} - \not{P}_{\mu\nu}(\partial h,\partial h) &= \Box_0 h^1_{\mu\nu} - \tilde{\Box}_g h^1_{\mu\nu} + \tilde{\Box}_g h^1_{\mu\nu} - \not{P}_{\mu\nu}(\partial h,\partial h)\\ &= (\Box_0-\tilde{\Box}_g)h^1_{\mu\nu} + \tilde{\Box}_g h_{\mu\nu} - \not{P}_{\mu\nu}(\partial h,\partial h) - \tilde{\Box}_g h^0_{\mu\nu}\end{aligned}$$

Using Lemma 9.14, (9.8), (9.10),(9.22) Lemma 9.12 and since $\tilde{\Box}_g h^0_{\mu\nu} = \Box_0 h^0_{\mu\nu}$ when $|x| \geq ct$ we get

$$\begin{aligned}|\Box_0 h^1_{\mu\nu} - \not{P}_{\mu\nu}(\partial h,\partial h)| \lesssim{}& \frac{\epsilon(1+q_+)^{-\gamma}}{(1+t+r)^{1+\gamma-2\delta}(1+|q|)^{1-\gamma}}\frac{C\epsilon(1+t)^{2\delta}}{(1+t+r)(1+q_+)^\gamma} + \frac{\epsilon^2}{(1+t+r)^{3-4\delta}(1+|q|)(1+q_+)^{4\delta}}\\ &+\frac{\epsilon}{(1+t)^3}+\frac{\epsilon M}{(1+t+r)^{4-2\delta}(1+q_+)^\gamma}\end{aligned}$$

Finally we get

$$|\Box_0 h^1_{\mu\nu} - \not{P}_{\mu\nu}(\partial h,\partial h)| \lesssim \frac{\epsilon^2}{(1+t+r)^{2+\gamma-4\delta}(1+|q|)^{2-\gamma}(1+q_+)^{4\delta}}$$

and the proposition follows. □

Now we shall show some estimates on first order derivatives.

Lemma 9.15. *Let $D_t = \{(t,x),\ |t-|x|| \leq c_0 t\}$, for some constant $0 < c_0 < 1$ and let $\overline{w}(q)$ be an increasing positive weight $\overline{w}'(q) \geq 0$. Then*

$$\begin{aligned}(1+t+|x|)|\partial\phi_{UV}(t,x)\overline{w}(q)| \lesssim{}& \sup_{0\leq\tau\leq t}\sum_{|I|\leq 1}\|Z^I\phi(\tau,.)\overline{w}\|_{L^\infty}\\ &+\int_0^t\Big((1+\tau)\|(\Box_0\phi)_{UV}(\tau,.)\overline{w}\|_{L^\infty(D_\tau)}+\sum_{|I|\leq 2}(1+\tau)^{-1}\|Z^I\phi(\tau,.)\overline{w}\|_{L^\infty(D_\tau)}\Big)d\tau\end{aligned} \tag{9.24}$$

Proof. Recall that the Laplacian in spherical coordinates takes the form

$$\Delta f = \frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\Big(\frac{\partial^2 f}{\partial\theta^2}+\frac{1}{tan(\theta)}\frac{\partial f}{\partial\theta}+\frac{1}{sin^2(\theta)}\frac{\partial^2 f}{\partial\phi^2}\Big)$$

Hence with $\Delta_\omega = \sum\Omega_{ij}^2$ we show :

$$\Box\phi = -\frac{1}{r}\big(\partial_t^2-\partial_r^2\big)(r\phi)+\frac{1}{r^2}\Delta_\omega\phi$$

Indeed, we have $\Omega_{ij}^2 = x_i^2\partial_{x_j}^2 - x_j^2\partial_{x_i}^2 - x^i\partial_{x_j}\partial_{x_i} + x^j\partial_{x_j}\partial_{x_i}$ for $i,j=1,2,3$ and in spherical coordinates :

$$\frac{\partial^2}{\partial x^2} = \Big(sin(\theta)cos(\phi)\frac{\partial}{\partial r}+\frac{cos(\theta)cos(\phi)}{r}\frac{\partial}{\partial\theta}-\frac{sin(\phi)}{rsin(\theta)}\frac{\partial}{\partial\phi}\Big)^2$$

$$\frac{\partial^2}{\partial y^2} = \Big(sin(\theta)sin(\phi)\frac{\partial}{\partial r}+\frac{cos(\theta)sin(\phi)}{r}\frac{\partial}{\partial\theta}+\frac{cos(\phi)}{rsin(\theta)}\frac{\partial}{\partial\phi}\Big)^2$$

$$\frac{\partial^2}{\partial z^2} = \Big(cos(\theta)\frac{\partial}{\partial r}-\frac{sin(\theta)}{r}\frac{\partial}{\partial\theta}\Big)^2$$

The result on the wave operator follows.

Now we have

$$(\Box_0-\Box)\phi = \frac{1}{r}\Big(-\frac{M}{r}\chi\Big(\frac{r}{1+t}\Big)(\partial_t^2+\partial_r^2)\Big)(r\phi)+\Big(-\frac{M}{r}\chi\Big(\frac{r}{1+t}\Big)\Big)\frac{1}{r^2}\Delta_\omega\phi$$

And from $ZU\le C$ for U in the null frame we have :

$$|\Box_0\phi_{UV}-U^\mu V^\nu\Box_0\phi_{\mu\nu}|\le r^{-2}\sum_{|J|\le 1}|Z^J\phi| \tag{9.25}$$

Now from (9.20), writing $\partial_q=\frac{\partial_r-\partial_t}{2}$ and $\partial_s=\frac{\partial_r+\partial_t}{2}$ we get

$$\Box_0\phi = \frac{1}{r}\Big(4\partial_s\partial_q-\frac{2M}{r}\chi\Big(\frac{r}{1+t}\Big)(\partial_q^2+\partial_s^2)\Big)(r\phi)+\Big(1-\frac{M}{r}\chi\Big(\frac{r}{1+t}\Big)\Big)\frac{1}{r^2}\Delta_\omega\phi$$

A straightforward computation shows that

$$\Big|\big(4\partial_s-2\frac{M}{r}\partial_q\big)\partial_q(r\phi)-r\Box_0\phi\Big|\lesssim r^{-1}\sum_{|J|\le 2}|Z^J\phi|$$

Hence for $|t-r|\le c_0t$, with $s=t+r$ and using (9.25) :

$$\Big|\big(\partial_s-\frac{M}{s}\partial_q\big)\partial_q(r\phi_{UV})\Big|\lesssim r|(\Box_0\phi)_{UV}|+(t+r)^{-1}\sum_{|J|\le 2}|Z^J\phi|$$

Multiplying by the weight $\overline{w}(q)$

$$\Big|\big(\partial_s-\frac{M}{s}\partial_q\big)\overline{w}(q)\partial_q(r\phi_{UV})\Big|\lesssim r|(\Box_0\phi)_{UV}|\overline{w}(q)+(t+r)^{-1}\sum_{|J|\le 2}|Z^J\phi|\overline{w}(q)$$

Let $(\tau,x(\tau))$ be an integral curve of the vector field $(\partial_s-\frac{M}{s}\partial_q)$ from the boundary of $D=\cup_{\tau\ge 0}D_\tau$ to any point inside D.
Let $\Psi:=\overline{w}(q)\partial_q(r\phi)$. Then

$$\Big|\frac{\partial}{\partial t}\Psi\Big|\le\big|\big(1-\frac{M}{s}\partial_q\big)\Psi\big|+\big|\big(\partial_s-\frac{M}{s}\partial_q\big)\Psi\big|$$

Using (9.7) and the fact that $\overline{w}'(q)\le C\frac{\overline{w}(q)}{1+|q|}$ we get

$$\Big|\frac{\partial}{\partial t}\Psi\Big|\lesssim\frac{1}{1+t}|\Psi|+r|(\Box_0\phi)_{UV}|\overline{w}(q)+(t+r)^{-1}\sum_{|J|\le 2}|Z^J\phi|\overline{w}(q)$$

(9.7) imply that we have for some (t_0,x_0) :

$$|\Psi(t_0,x_0)|\le C\sum_{|I|\le 1}\overline{w}(q)|Z^I\phi|$$

Using the Grönwall inequality we get :

$$\begin{aligned}|\partial_q(r\phi_{UV}(t,x))\overline{w}(q)|\lesssim &\sup_{0\le\tau\le t}\sum_{|I|\le 1}\|Z^I\phi(\tau,.)\overline{w}\|_{L^\infty}\\ &+\int_0^t\Big((1+\tau)\|(\Box_0\phi)_{UV}(\tau,.)\overline{w}\|_{L^\infty(D_\tau)}+\sum_{|I|\le 2}(1+\tau)^{-1}\|Z^I\phi(\tau,.)\overline{w}\|_{L^\infty(D_\tau)}\Big)d\tau\end{aligned}$$

The lemma now follows from (9.6), that the estimate is true for h and when $|r-t|\ge c_0t$.

$\square$

Lemma 9.16. *Let $D_t = \{(t,x),\ |t-|x|| \le c_0 t\}$, for some constant $0 < c_0 < 1$ and let $\overline{w}(q) = (1+q_+)^{1+\gamma'}$, where $-1 \le \gamma' < \gamma - 2\delta$. Then*

$$(1+t+|x|)|\partial h^1_{UV}(t,x)\overline{w}(q)| \lesssim \epsilon + \int_0^t (1+\tau)\|(\Box_0 h^1)_{UV}(\tau,.)\overline{w}\|_{L^\infty(D_\tau)}d\tau$$

Proof. The proof follows from the previous lemma and the bound (9.8). □

Proposition 9.6. *If the weak energy bounds and initial bounds hold then we have for any $0 \le \gamma' < \gamma - 4\delta$*

$$(1+t+r)(1+q_+)^{1+\gamma'}|\partial h^1_{TU}| \lesssim \epsilon \tag{9.26}$$

$$(1+t+r)(1+q_+)^{1+\gamma'}|\partial h^1| \lesssim \epsilon(1+\epsilon ln(2+t)) \lesssim \epsilon(1+t)^{\epsilon}. \tag{9.27}$$

The same estimates hold for h in place of h^1 if $\gamma' = 0$

Proof. For (6.31) we note that $\not{P}_{UV} = 0$ in D_t. For $t \ge 2K/(1-c)$ we also have $\hat{T} = 0$ in D_t if we pick c_0 small enough. For $0 \le t \le 2K/(1-c)$ we have $\hat{T}(t,x)$ uniformly bounded with (9.10). The previous Lemma and Proposition 9.5 give (9.26). On order to have (9.27) we note that the only new term is $\not{P}_{\mu\nu}$ which is controlled by

$$|\not{P}(\partial h, \partial h)| \le |\partial h_{TS}|^2 \lesssim \epsilon^2(1+t+r)^{-2}(1+q_+)^{-2-2\gamma'}$$

Multiplying by $(1+t)$, integrating, and using Lemma 9.16 and Proposition 9.5 gives the result.

□

9.5 Lie derivatives

Before using commutators and Lie derivatives for our problem, we introduce in this subsection the concept behind Lie derivative. The idea behind the Lie derivatives acting on a function $f : \mathcal{M} \to \mathbb{R}$ is to quantify how much the function f changes along the flow of a given vector field Z. In fact, a Lie derivative can be introduced only with a vector field Z. For any vector field Z there exists a smooth map $\sigma : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ called the flow of Z such that for $t \in \mathbb{R}$, and $p \in \mathcal{M}$ we write $\sigma_t(p)$ such that

$$\begin{aligned}\sigma_0(p) &= p\\ \sigma_t(\sigma_s(p)) &= \sigma_{t+s}(p)\\ \frac{d}{dt}\sigma_t &= Z(\sigma_t(p))\end{aligned}$$

In coordinate x^μ we can see that

$$\sigma^\mu_t(p) = x^\mu(p) + tZ^\mu(p) + O(t^2)$$

when t goes to 0. Hence the flow along Z when t is small enough goes in the direction of Z.
This description implies that we want to define our Lie derivative applied to f for the vector field Z as

$$\mathcal{L}_Z f(p) = \lim_{t\to 0}\frac{1}{t}[f(\sigma_t(p)) - f(p)]$$

for $p \in \mathcal{M}$. Using

$$f(\sigma_t(p)) = f(p) + tZ^\mu\frac{\partial f}{\partial x^\mu}(p) + O(t^2)$$

we get

$$\mathcal{L}_Z f = Z^\mu\frac{\partial f}{\partial x^\mu}$$

The Lie derivative applied for a vector field W is a bit different. Indeed $W(\sigma_t(p))$ and $W(p)$ belong to two different tangent spaces and we can't take the limit when t goes to zero of the difference. But this can be done with the induced map $D_{\sigma_t(p)}\sigma_{-t} : T_{\sigma_t(p)}(\mathcal{M}) \to T_p(\mathcal{M})$. Hence the Lie derivative of the vector field is defined as :

$$\mathcal{L}_Z W(p) = \lim_{t\to 0}\frac{1}{t}[D_{\sigma_t(p)}\sigma_{-t}(W) - W(p)]$$

And in appropriate coordinates it follows that

$$(\mathcal{L}_Z W)^\mu = Z^\nu\frac{\partial W^\mu}{\partial x^\nu} - W^\nu\frac{\partial V^\mu}{\partial x^\nu}$$

Hence

$$\mathcal{L}_Z W = [V, W]$$

Now we generalize the previous equation to (r,s). The Lie derivative applied to a (r,s) tensor K is defined by

$$\mathcal{L}_Z K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} = ZK^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} - \partial_\gamma Z^{\alpha_1}K^{\gamma\ldots\alpha_r}_{\beta_1\ldots\beta_s} - \ldots - \partial_\gamma Z^{\alpha_r}K^{\alpha_1\ldots\gamma}_{\beta_1\ldots\beta_s} + \partial_{\beta_1}Z^\gamma K^{\alpha_1\ldots\alpha_r}_{\gamma\ldots\beta_s} + \ldots + \partial_{\beta_s}Z^\gamma K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\gamma}$$

9.6 The commutators and Lie derivatives

We want to apply the energy inequality

$$\int_{\Sigma_t} |\partial\phi|^2 w dx + \int_0^t \int_{\Sigma_\tau} |\overline{\partial}\phi|^2 w' dx d\tau \leq 8 \int_{\Sigma_0} |\partial\phi|^2 w dx + \int_0^t \frac{C\epsilon}{1+\tau} \int_{\Sigma_t} |\partial\phi|^2 w dx d\tau$$
$$+ 16 \int_0^t \Big(\int_{\Sigma_\tau} |\tilde{\Box}_g \phi|^2 w dx \Big)^{1/2} \Big(|\partial\phi|^2 w dx \Big)^{1/2} d\tau$$

. In order to apply it, we need to commute the system with the vector fields. Instead of commuting with Z directly we will commute the system with Lie derivatives $\mathcal{L}_Z$ that appear in

$$\tilde{\Box}_g Z\phi = Z(\tilde{\Box}_g \phi) - (\mathcal{L}_Z g^{\alpha\beta})\partial_\alpha \partial_\beta \phi$$

The Lie derivatives will simplify the commutators by removing the lower order terms.
The Lie derivative applied to a (r,s) tensor K is defined by

$$\mathcal{L}_Z K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} = Z K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} - \partial_\gamma Z^{\alpha_1} K^{\gamma\ldots\alpha_r}_{\beta_1\ldots\beta_s} - \ldots - \partial_\gamma Z^{\alpha_r} K^{\alpha_1\ldots\gamma}_{\beta_1\ldots\beta_s} + \partial_{\beta_1} Z^\gamma K^{\alpha_1\ldots\alpha_r}_{\gamma\ldots\beta_s} + \ldots + \partial_{\beta_s} Z^\gamma K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\gamma} \tag{9.28}$$

Lie derivative satisfies Leibnitz rule. For the case of our vector fields $\partial_\gamma Z^\beta$ are constant, which results in the following commutation properties.

Proposition 9.7. *If K is an (r,s) tensor the, with respect to the coordinate system $\{x^\mu\}$, the vector fields $Z = \partial_{x^\mu}, \Omega_{ij}, B_i, S$ satisfy,*

$$\mathcal{L}_Z \partial_{\mu_1}\ldots\partial_{\mu_k} K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} = \partial_{\mu_1}\ldots\partial_{\mu_k} \mathcal{L}_Z K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} \tag{9.29}$$

and

$$\mathcal{L}_Z \partial_\mu K^{\mu\ldots\alpha_r}_{\beta_1\ldots\beta_s} = \partial_\mu \mathcal{L}_Z K^{\mu\ldots\alpha_r}_{\beta_1\ldots\beta_s} \tag{9.30}$$

Proof. Let $\hat{K}^{\alpha_1\ldots\alpha_r}_{\mu_1\ldots\mu_k\beta_1\ldots\beta_s} = \partial_{\mu_1}\ldots\partial_{\mu_k}\mathcal{L}_Z K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}$ Then using the definition (9.28) we have

$$\mathcal{L}_Z \hat{K}^{\alpha_1\ldots\alpha_r}_{\mu_1\ldots\mu_k\beta_1\ldots\beta_s} = Z\big(\partial_{\mu_1}\ldots\partial_{\mu_k} K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}\big) - \partial_\gamma Z^{\alpha_1}\partial_{\mu_1}\ldots\partial_{\mu_k} K^{\gamma\ldots\alpha_r}_{\beta_1\ldots\beta_s} - \ldots - \partial_\gamma Z^{\alpha_r}\partial_{\mu_1}\ldots\partial_{\mu_k} K^{\alpha_1\ldots\gamma}_{\beta_1\ldots\beta_s}$$
$$+ \partial_{\mu_1} Z^\gamma \partial_\gamma\ldots\partial_{\mu_k} K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} + \ldots + \partial_{\mu_k} Z^\gamma \partial_{\mu_1}\ldots\partial_\gamma K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}$$
$$+ \partial_{\beta_1} Z^\gamma \partial_{\mu_1}\ldots\partial_{\mu_k} K^{\alpha_1\ldots\alpha_r}_{\gamma\ldots\beta_s} + \ldots + \partial_{\beta_s} Z^\gamma \partial_{\mu_1}\ldots\partial_{\mu_k} K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\gamma}$$

Now, we have

$$\partial_{\mu_1}\ldots\partial_{\mu_k}\mathcal{L}_Z K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} = \partial_{\mu_1}\ldots\partial_{\mu_k}\Big(Z^\gamma\partial_\gamma\big(K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}\big) - \partial_\gamma Z^{\alpha_1} K^{\gamma\ldots\alpha_r}_{\beta_1\ldots\beta_s} - \ldots - \partial_\gamma Z^{\alpha_r} K^{\alpha_1\ldots\gamma}_{\beta_1\ldots\beta_s}$$
$$+ \partial_{\beta_1} Z^\gamma K^{\alpha_1\ldots\alpha_r}_{\gamma\ldots\beta_s} + \ldots + \partial_{\beta_s} Z^\gamma K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\gamma}\Big)$$

Since $\partial_{x^\alpha} Z^\beta$ is constant it remains to show that

$$\partial_{\mu_1}\ldots\partial_{\mu_k} Z^\gamma\partial_\gamma\big(K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}\big) = Z\big(\partial_{\mu_1}\ldots\partial_{\mu_k} K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}\big) + \partial_{\mu_1} Z^\gamma\partial_\gamma\ldots\partial_{\mu_k} K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} + \ldots + \partial_{\mu_k} Z^\gamma \partial_{\mu_1}\ldots\partial_\gamma K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}$$

This can be shown using the fact that since $\partial_{x^\alpha} Z^\beta$ is constant we have

$$\partial_{\mu_1}\ldots\partial_{\mu_k} Z^\gamma\partial_\gamma\big(K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}\big) = \partial_{\mu_1}\ldots\partial_{\mu_{k-1}}(\partial_{\mu_k} Z^\gamma)\partial_\gamma\big(K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}\big) + \partial_{\mu_1}\ldots\partial_{\mu_{k-1}} Z^\gamma\partial_\gamma\partial_{\mu_k}\big(K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}\big)$$
$$= \partial_{\mu_1}\ldots\partial_{\mu_{k-2}}(\partial_{\mu_k} Z^\gamma)\partial_\gamma\partial_{\mu_{k-1}}\big(K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}\big) + \partial_{\mu_1}\ldots\partial_{\mu_{k-2}}(\partial_{\mu_{k-1}} Z^\gamma)\partial_\gamma\partial_{\mu_k}\big(K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}\big) + \partial_{\mu_1}\ldots\partial_{\mu_{k-2}} Z^\gamma\partial_\gamma\partial_{\mu_k}\partial_{\mu_{k-1}}\big(K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}\big)$$

Iterating this process gives (9.29). (9.30) follows directly from (9.29).

□

We now define the modified Lie derivative by

$$\widehat{\mathcal{L}}_Z K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} = \mathcal{L}_Z K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} + \frac{r-s}{4}(\partial_\gamma Z^\gamma) K^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} \tag{9.31}$$

From this definition it is straightforward to show that we get $\widehat{\mathcal{L}}_Z m^{\alpha\beta} = 0$ and $\widehat{\mathcal{L}}_Z m_{\alpha\beta} = 0$ for the vector fields in our collection.
Now let $h_{\alpha\beta}$ and $k_{\alpha\beta}$ be two (0,2) tensors and let $S_{\mu\nu}(\partial h, \partial k)$ be a (0,2) tensor which is a quadratic form in the (0,3) tensors ∂h and ∂k with two contractions with the Minkowski metric (in particular

$P_{\mu\nu}(\partial_\mu h, \partial_\nu h)$ and $Q_{\mu\nu}(\partial h, \partial k)$). Then, using the fact the the Lie derivative satisfy the Leibnitz rule and using the previous proposition we have

$$\mathcal{L}_Z(S_{\mu\nu}(\partial h, \partial k)) = S_{\mu\nu}(\partial \widehat{\mathcal{L}}_Z h, \partial k) + S_{\mu\nu}(\partial h, \partial \widehat{\mathcal{L}}_Z k) \tag{9.32}$$

Moreover since $\widehat{\mathcal{L}}_Z m^{\alpha\beta} = 0$ and $\widehat{\mathcal{L}}_Z m_{\alpha\beta} = 0$

$$\widehat{\mathcal{L}}_Z g^{\alpha\beta} = \mathcal{L}_Z h^{\alpha\beta} - \frac{1}{2}(\partial_\gamma Z^\gamma) h^{\alpha\beta} \qquad \widehat{\mathcal{L}}_Z h_{\mu\nu} = \mathcal{L}_Z h_{\mu\nu} + \frac{1}{2}(\partial_\gamma Z^\gamma) h_{\mu\nu}$$

Using the fact that

$$H^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu} = -h^{\mu\nu} + O^{\mu\nu}(h^2) \quad \text{where} \quad h^{\mu\nu} = m^{\mu\mu'} m^{\nu\nu'} h_{\mu'\nu'}$$

we get

$$\mathcal{L}_Z(g^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}) = (\widehat{\mathcal{L}}_Z g^{\alpha\beta})\partial_\alpha\partial_\beta h_{\mu\nu} + g^{\alpha\beta}\partial_\alpha\partial_\beta \widehat{\mathcal{L}}_Z h_{\mu\nu} \tag{9.33}$$

Now let $\mathcal{L}_Z^I$ be a product of $|I|$ Lie derivatives with respect to $|I|$ vector fields Z.
Recall that under the harmonic gauge, it has been shown that the Einstein-Vlasov system takes the form :

$$\widetilde{\Box}_g h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h) + \widehat{T}_{\mu\nu} \qquad \text{where} \qquad \widetilde{\Box}_g = (g^{\alpha\beta})\partial_\alpha\partial_\beta$$

and $\widehat{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} tr_g T$, $\quad tr_g T = g^{ij} T_{ij}$ $\quad$ and $F_{\mu\nu}(u)(v,v)$ depends quadratically on v.
Hence we have

Proposition 9.8.

$$\widetilde{\Box}_g \widehat{\mathcal{L}}_Z^I h_{\mu\nu} = [\widetilde{\Box}_g \widehat{\mathcal{L}}_Z^I - \mathcal{L}_Z^I \widetilde{\Box}_g] h_{\mu\nu} + \mathcal{L}_Z^I F_{\mu\nu}(H)(\partial h, \partial h) + \mathcal{L}_Z^I T_{\mu\nu}$$

where

$$[\widetilde{\Box}_g \widehat{\mathcal{L}}_Z^I - \mathcal{L}_Z^I \widetilde{\Box}_g]\phi_{\mu\nu} = - \sum_{J+K=I, |K|<|I|} \widehat{\mathcal{L}}_Z^J H^{\alpha\beta}\partial_\alpha\partial_\beta \widehat{\mathcal{L}}_Z^K \phi_{\mu\nu}$$

and

$$\mathcal{L}_Z^I F_{\mu\nu}(H)(\partial h, \partial h) = \sum_{J+K=I} P\big(\partial_\mu \widehat{\mathcal{L}}_Z^J h, \partial_\nu \widehat{\mathcal{L}}_Z^K h\big) + \sum_{J+K=I} Q\big(\partial \widehat{\mathcal{L}}_Z^J h, \partial \widehat{\mathcal{L}}_Z^K h\big) + \mathcal{L}_Z^I G_{\mu\nu}(H)(\partial h, \partial h),$$

where

$$|\mathcal{L}_Z^I G_{\mu\nu}(H)(\partial h, \partial h)| \lesssim \sum_{I_1+I_2+\ldots+I_k=I, k\geq 3} |\widehat{\mathcal{L}}_Z^{I_3} H| \cdots |\widehat{\mathcal{L}}_Z^{I_k} H||\partial \widehat{\mathcal{L}}_Z^{I_1} h||\partial \widehat{\mathcal{L}}_Z^{I_2} h|$$

Proof. We have

$$\mathcal{L}_Z^I(\widetilde{\Box}_g h_{\mu\nu}) = \widetilde{\Box}_g \widehat{\mathcal{L}}_Z^I h_{\mu\nu} + \sum_{J+K=I, |K|<I} (\widehat{\mathcal{L}}_Z^J g^{\alpha\beta}\partial_\alpha\partial_\beta \widehat{\mathcal{L}}_Z^K h_{\mu\nu})$$

which results from a straightforward induction on (9.33) and with the previous sytem

$$\mathcal{L}_Z^I(\widetilde{\Box}_g h_{\mu\nu}) = \mathcal{L}_Z^I\big(F_{\mu\nu}(h)(\partial h, \partial h)\big) + \mathcal{L}_Z^I(\widehat{T}_{\mu\nu})$$

Hence

$$\widetilde{\Box}_g \widehat{\mathcal{L}}_Z^I h_{\mu\nu} = \mathcal{L}_Z^I\big(F_{\mu\nu}(h)(\partial h, \partial h)\big) + \mathcal{L}_Z^I(\widehat{T}_{\mu\nu}) - \sum_{J+K=I, |K|<I} (\widehat{\mathcal{L}}_Z^J H^{\alpha\beta}\partial_\alpha\partial_\beta \widehat{\mathcal{L}}_Z^K h_{\mu\nu})$$

Finally, (9.33) gives the result

□

With Proposition 9.7 and from the wave coordinate equation we now have

$$\partial_\mu \widehat{\mathcal{L}}_Z \widehat{H}^{\mu\nu} = (\widehat{\mathcal{L}}_Z + \frac{\partial_\gamma Z^\gamma}{2}) W^\nu(H, \partial h)$$

and writing

$$\widehat{\mathcal{L}}_Z^I \widehat{H}^{\mu\nu} = \widehat{\mathcal{L}}_Z^I H^{\mu\nu} - m^{\mu\nu} tr_m \widehat{\mathcal{L}}_Z^I H/2 \qquad tr \widehat{\mathcal{L}}_Z^I H = m_{\alpha\beta}\widehat{\mathcal{L}}_Z^I H^{\alpha\beta}$$

it follows

$$|\partial_\mu \widehat{\mathcal{L}}_Z^I \widehat{H}^{\mu\nu}| \lesssim \sum_{I_1+\ldots+I_k} |\widehat{\mathcal{L}}_Z^{I_2} H| \cdots |\widehat{\mathcal{L}}_Z^{I_k} H||\partial \widehat{\mathcal{L}}_Z^{I_1} h| \tag{9.34}$$

Finally we have the following estimates with the definition of Lie derivatives which only add lower order terms :

$$\sum_{|I|\leq k} |Z^I K| \lesssim \sum_{|I|\leq k} |\widehat{\mathcal{L}}_Z^I K| \lesssim \sum_{|I|\leq k} |Z^I K| \tag{9.35}$$

Lemma 9.17. *We have*

$$|\partial_q \widehat{\mathcal{L}}_Z^I H|_{LT} + |\partial_q \not{tr}\, \widehat{\mathcal{L}}_Z^I H| \lesssim |\overline{\partial} \widehat{\mathcal{L}}_Z^I H| + \sum_{|J|+|K|\le |I|} |\widehat{\mathcal{L}}_Z^J h||\partial \widehat{\mathcal{L}}_Z^K h|, \tag{9.36}$$

$$|\partial_q \widehat{\mathcal{L}}_Z^I H_1|_{LT} + |\partial_q \not{tr}\, \widehat{\mathcal{L}}_Z^I H_1| \lesssim |\overline{\partial} \widehat{\mathcal{L}}_Z^I H_1| + \sum_{|J|+|K|\le |I|} |\widehat{\mathcal{L}}_Z^J h||\partial \widehat{\mathcal{L}}_Z^K h| + M\big|\chi'(\frac{r}{t+1})\big|(1+t+r)^{-2}, \tag{9.37}$$

where $\chi'(s)$, is a function supported when $1/4 \le s \le 1/2$ and $T = \{L, S_1, S_2\}$. Moreover, (9.36) hold also for h in place of H.

Proof. The proof follows the proof from Lemma 9.10, using the estimate (9.34) and the fact that $|\widehat{\mathcal{L}}_Z^{I_k} h| \lesssim 1$ for small $|I_k|$ (see 9.35). □

Proposition 9.9. *For $|I| \le N-4$ we have*

$$|\partial_q \widehat{\mathcal{L}}_Z^I H|_{LT} + |\partial_q \not{tr}\, \widehat{\mathcal{L}}_Z^I H| \lesssim \epsilon (1+t+r)^{-2+2\delta}(1+q_+)^{-\gamma}, \tag{9.38}$$

$$|\widehat{\mathcal{L}}_Z^I H_1|_{LT} + |\not{tr}\, \widehat{\mathcal{L}}_Z^I H_1| \lesssim \epsilon(1+t+r)^{-1-\gamma+2\delta} + \epsilon(1+t)^{-2+2\delta}(1+q_-) \lesssim \epsilon(1+t+r)^{-1-\gamma+2\delta}(1+q_-)^{\gamma} \tag{9.39}$$

The same estimates hold for H in place of H_1 if γ is replaced by 2δ.

Proof. The proof is the same as in Proposition 9.3 □

Now that we used the wave coordinate equation to get some bounds, we shall give estimates for the inhomogeneous term, the wave operator applied to h^0 and the commutator term.
Recall that

$$|\mathcal{L}_Z^I G_{\mu\nu}(H)(\partial h, \partial h)| \lesssim \sum_{I_1+I_2+\ldots+I_k = I, k\ge 3} |\widehat{\mathcal{L}}_Z^{I_3} H| \cdots |\widehat{\mathcal{L}}_Z^{I_k} H||\partial \widehat{\mathcal{L}}_Z^{I_1} h||\partial \widehat{\mathcal{L}}_Z^{I_2} h|$$

Now using the fact that $|\widehat{\mathcal{L}}_Z^{I_k} h| \lesssim 1$ for small $|I_k|$ we get

$$|\mathcal{L}_Z^I G_{\mu\nu}(H)(\partial h, \partial h)| \lesssim \sum_{|I_1|+|I_2|+|I_3|\le |I|} |\widehat{\mathcal{L}}_Z^{I_3} H||\partial \widehat{\mathcal{L}}_Z^{I_1} h||\partial \widehat{\mathcal{L}}_Z^{I_2} h|$$

And for any term satisfying classical null condition we have according to the estimates $|Q_{\mu\nu}(\partial h, \partial k)| \lesssim |\overline{\partial} h||\partial k| + |\partial h||\overline{\partial} k|$:

$$\Big|\sum_{J+K=I} Q\big(\partial \widehat{\mathcal{L}}_Z^J h, \partial \widehat{\mathcal{L}}_Z^K h\big)\Big| \lesssim \sum_{|J|+|K|\le |I|} |\overline{\partial} \widehat{\mathcal{L}}_Z^J h||\partial \widehat{\mathcal{L}}_Z^K h|.$$

Recall the expression of $\not{P}_{\mu\nu}$ from (9.19). With (9.16) and (9.18) we get

$$\sum_{J+K=I} \big|P\big(\partial_\mu \widehat{\mathcal{L}}_Z^J h, \partial_\nu \widehat{\mathcal{L}}_Z^K h\big)\big| \lesssim \sum_{|J|+|K|\le |I|} \big|\not{P}_{\mu\nu}\big(\partial \widehat{\mathcal{L}}_Z^J h, \partial \widehat{\mathcal{L}}_Z^K h\big)\big| + \sum_{|J|+|K|\le |I|} \big(\big|\partial_q \widehat{\mathcal{L}}_Z^J h\big|_{L\mathcal{T}} + \big|\partial_q \not{tr}\, \widehat{\mathcal{L}}_Z^J h\big|\big)\big|\partial \widehat{\mathcal{L}}_Z^K h\big|$$

Finally we get

$$\begin{aligned}
&|\mathcal{L}_Z^I F_{\mu\nu}(h)(\partial h, \partial h)| \\
&\lesssim \sum_{|J|+|K|\le |I|} \big|\not{P}_{\mu\nu}\big(\partial \widehat{\mathcal{L}}_Z^J h, \partial \widehat{\mathcal{L}}_Z^K h\big)\big| + \sum_{|J|+|K|\le |I|} |\overline{\partial} \widehat{\mathcal{L}}_Z^J h||\partial \widehat{\mathcal{L}}_Z^K h| + \sum_{|I_1|+|I_2|+|I_3|\le |I|} |\widehat{\mathcal{L}}_Z^{I_3} h||\partial \widehat{\mathcal{L}}_Z^{I_1} h||\partial \widehat{\mathcal{L}}_Z^{I_2} h|
\end{aligned}$$

Using the expression (9.19) and splitting the sums between low and high derivatives we get

$$\begin{aligned}
&|\mathcal{L}_Z^I F_{\mu\nu}(h)(\partial h, \partial h)| \\
&\lesssim \big(|\partial h|_{SS} + |\overline{\partial} h| + |h||\partial h|\big) \sum_{|J|\le |I|} \big|\partial \widehat{\mathcal{L}}_Z^J h\big| + |\partial h| \sum_{|J|\le |I|} |\overline{\partial} \widehat{\mathcal{L}}_Z^J h| + |\partial h|^2 \sum_{|J|\le |I|} |\widehat{\mathcal{L}}_Z^J h| + \sum_{|K|\le |I|/2} \big|\partial \widehat{\mathcal{L}}_Z^K h\big| \sum_{|J|\le |I|-1} \big|\partial \widehat{\mathcal{L}}_Z^J h\big| \\
&\lesssim \frac{\epsilon(1+q_+)^{-1}}{1+t+r} \sum_{|J|\le |I|} \big|\partial \widehat{\mathcal{L}}_Z^J h\big| + \frac{\epsilon(1+t)^{2\delta}(1+q_+)^{-2\delta}}{(1+t+r)(1+|q|)} \sum_{|J|\le |I|} |\overline{\partial} \widehat{\mathcal{L}}_Z^J h| + \frac{\epsilon^2(1+t)^{4\delta}(1+q_+)^{-4\delta}}{(1+t+r)^2(1+|q|)^2} \sum_{|J|\le |I|} |\widehat{\mathcal{L}}_Z^J h| \\
&\qquad + \sum_{|K|\le |I|/2} \big|\partial \widehat{\mathcal{L}}_Z^K h\big| \sum_{|J|\le |I|-1} \big|\partial \widehat{\mathcal{L}}_Z^J h\big|
\end{aligned} \tag{9.40}$$

We now give estimates of the wave operator applied to h^0. Recall from (9.22) that with the definition of $h^0_{\mu\nu}$, there exists smooth cut-off functions $\chi_{\alpha\beta}(s,\omega)$ supported when $s \geq 1/4$ such that

$$|(\widetilde{\square}_g - \square_0)h^0_{\mu\nu}| = \frac{M\delta_{\mu\nu}}{(1+t+r)^3} H_1^{\alpha\beta}\chi_{\alpha\beta}\Big(\frac{r}{t+1}, w\Big)$$

Hence

$$\big|\mathcal{L}_Z^I(\widetilde{\square}_g - \square_0)h^0_{\mu\nu}\big| \lesssim \frac{M}{(1+t+r)^3}\sum_{|J|\leq|I|}|\widehat{\mathcal{L}}_Z^J H_1| \tag{9.41}$$

and using (9.21) we get

$$\big|\mathcal{L}_Z^I\square_0 h^0_{\mu\nu}\big| \lesssim \frac{M}{(1+t+r)^3}\chi'(\frac{r}{1+t}) \tag{9.42}$$

where $\chi'(s)$ is supported in $1/4 \leq s \leq 1$.

It remains to estimate the term $[\widetilde{\square}_g\widehat{\mathcal{L}}_Z^I - \mathcal{L}_Z^I\widetilde{\square}_g]\phi_{\mu\nu}$. Recall that we have the following expression for this term

$$[\widetilde{\square}_g\widehat{\mathcal{L}}_Z^I - \mathcal{L}_Z^I\widetilde{\square}_g]\phi_{\mu\nu} = -\sum_{J+K=I,|K|<|I|}\widehat{\mathcal{L}}_Z^J H^{\alpha\beta}\partial_\alpha\partial_\beta\widehat{\mathcal{L}}_Z^K\phi_{\mu\nu}$$

Using Lemma 9.13 we can get a bound for this term and we have

$$\big|[\widetilde{\square}_g\widehat{\mathcal{L}}_Z^I - \mathcal{L}_Z^I\widetilde{\square}_g]\phi_{\mu\nu}\big| \lesssim \sum_{\substack{|J|+|K|-1\leq|I|\\1\leq|K|\leq|I|}}\Big(\frac{\big|(\widehat{\mathcal{L}}_Z^J H)_{LL}\big|}{1+|q|} + \frac{\big|\widehat{\mathcal{L}}_Z^J H\big|}{1+t+r}\Big)\big|\partial\widehat{\mathcal{L}}_Z^K\phi_{\mu\nu}\big| \tag{9.43}$$

Now, since we have $\widetilde{\square}_g = \square_0 + (H^{\alpha\beta} - H_0^{\alpha\beta})\partial_\alpha\partial_\beta$ we get

$$[\square_0\widehat{\mathcal{L}}_Z^I - \mathcal{L}_Z^I\square_0]\phi_{\mu\nu} = -\sum_{J+K=I,|K|<|I|}\widehat{\mathcal{L}}_Z^J H_0^{\alpha\beta}\partial_\alpha\partial_\beta\widehat{\mathcal{L}}_Z^K\phi_{\mu\nu}$$

Using the fact that $H_0 \sim \frac{1}{r}$ we have in a similar way

$$\big|[\square_0\widehat{\mathcal{L}}_Z^I - \mathcal{L}_Z^I\square_0]\phi_{\mu\nu}\big| \lesssim \sum_{|J|\leq|I|}\Big(\frac{\big|(\widehat{\mathcal{L}}_Z^J H_0)_{LL}\big|}{1+|q|} + \frac{\big|\widehat{\mathcal{L}}_Z^J H_0\big|}{1+t+r}\Big)\sum_{|K|\leq|I|}\big|\partial\widehat{\mathcal{L}}_Z^K\phi_{\mu\nu}\big| \lesssim \frac{M(1+|q|)^{-1}}{1+t+r}\sum_{|K|\leq|I|}\big|\partial\widehat{\mathcal{L}}_Z^K\phi_{\mu\nu}\big| \tag{9.44}$$

Using proposition 9.9 and (9.8) we also get

$$\sum_{|J|\leq|I|/2+1}\Big(\frac{\big|(\widehat{\mathcal{L}}_Z^J H_1)_{LL}\big|}{1+|q|} + \frac{\big|\widehat{\mathcal{L}}_Z^J H_1\big|}{1+t+r}\Big)\sum_{|K|\leq|I|}\big|\partial\widehat{\mathcal{L}}_Z^K\phi_{\mu\nu}\big| \lesssim \frac{\epsilon(1+q_+)^{-\gamma}}{(1+t+r)^{1+\gamma-2\delta}(1+|q|)^{1-\gamma}}\sum_{|K|\leq|I|}\big|\partial\widehat{\mathcal{L}}_Z^K\phi_{\mu\nu}\big|$$

And finally we have

$$\begin{aligned}&\big|[(\widetilde{\square}_g - \square_0)\widehat{\mathcal{L}}_Z^I - \mathcal{L}_Z^I(\widetilde{\square}_g - \square_0)]\phi_{\mu\nu}\big| \\ &\lesssim \frac{\epsilon(1+q_+)^{-\gamma}}{(1+t+r)^{1+\gamma-2\delta}(1+|q|)^{1-\gamma}}\sum_{|K|\leq|I|}\big|\partial\widehat{\mathcal{L}}_Z^K\phi_{\mu\nu}\big| + \sum_{|K|\leq|I|/2}\big|\partial\widehat{\mathcal{L}}_Z^K\phi_{\mu\nu}\big| + \sum_{|J|\leq|I|}\Big(\frac{\big|(\widehat{\mathcal{L}}_Z^J H_1)_{LL}\big|}{1+|q|} + \frac{\big|\widehat{\mathcal{L}}_Z^J H_1\big|}{1+t}\Big)\end{aligned} \tag{9.45}$$

9.7 The sharp L^∞ decay estimates for higher low derivatives

In this following subsection, we do not change many things from the article of Hans Lindblad and Martin Taylor [1] (p. 62,63) since the computations are well explained. As they notice, "As we have alreadly proven the higher order weak decay estimates in Proposition 9.2 and the higher order sharp decay estimates for components we control with the wave condition in Porposition 9.9 it only remains to generalize proposition 9.6 to higher order. In order to do that we will as before rely on the crucial Lemma 9.15 to control transversal derivatives in terms of tangential derivatives, which w control by proposition 9.2, and $\square_0$ close to the light cone $|t-r| < 1-c$. It therefore only remains to get control of $\square_0\widehat{\mathcal{L}}_Z^I h^1_{\mu\nu}$ close to the light cone $|t-r| < (1-c)t$ where $0 < c < 1$. When $|t-r| < (1-c)t$ we have by (9.21) and (9.22) "

$$\mathcal{L}_Z^I\widetilde{\square}_g h^1_{\mu\nu} = \mathcal{L}_Z^I F_{\mu\nu} - \mathcal{L}_Z^I(\widetilde{\square}_g - \square_0)h^0_{\mu\nu}, \qquad |t-r| < (1-c)t$$

Recall (9.41) which gives an estimmate on the second term of the right hand side.

$$\big|\mathcal{L}_Z^I(\widetilde{\Box}_g-\Box_0)h^0_{\mu\nu}\big|\lesssim\frac{M}{(1+t+r)^3}\sum_{|J|\le|I|}\big|\widehat{\mathcal{L}}_Z^J H_1\big|$$

Using Proposition 9.2 we get

$$\big|\mathcal{L}_Z^I(\widetilde{\Box}_g-\Box_0)h^0_{\mu\nu}\big|\lesssim\frac{\epsilon M}{(1+t+r)^{3-2\delta}(1+q_+)^{\gamma}}$$

With (9.40) we also have

$$(9.46)\quad |\mathcal{L}_Z^I F_{\mu\nu}(H)(\partial h,\partial h)|$$
$$\lesssim\frac{\epsilon(1+q_+)^{-1}}{1+t+r}\sum_{|J|\le|I|}\big|\partial\widehat{\mathcal{L}}_Z^J h\big|+\frac{\epsilon^2(1+q_+)^{-4\delta}}{(1+t+r)^{3-4\delta}(1+|q|)}+\frac{\epsilon^3(1+q_+)^{-6\delta}}{(1+t+r)^{3-6\delta}(1+|q|)^3}+\sum_{|K|\le|I|/2}\big|\partial\widehat{\mathcal{L}}_Z^K h\big|\sum_{|J|\le|I|-1}\big|\partial\widehat{\mathcal{L}}_Z^J h\big|$$

Now

$$\Box_0\widehat{\mathcal{L}}_Z^I h^1_{\mu\nu}=\mathcal{L}_Z^I\Box_0 h^1_{\mu\nu}+[\Box_0\widehat{\mathcal{L}}_Z^I-\mathcal{L}_Z^I\Box_0]h^1_{\mu\nu}=\mathcal{L}_Z^I\widetilde{\Box}_g h^1_{\mu\nu}-\mathcal{L}_Z^I(\widetilde{\Box}_g-\Box_0)h^1_{\mu\nu}+[\Box_0\widehat{\mathcal{L}}_Z^I-\mathcal{L}_Z^I\Box_0]h^1_{\mu\nu}$$

We have already estimated the first term in the right hand side. It now remains to estimate the second and the third one. In order to have bounds on the third term we recall the (9.44) and we get

$$\big|[\Box_0\widehat{\mathcal{L}}_Z^I-\mathcal{L}_Z^I\Box_0]h^1_{\mu\nu}\big|\lesssim\frac{\epsilon(1+|q|)^{-1}}{1+t+r}\sum_{|K|\le|I|}\big|\partial\widehat{\mathcal{L}}_Z^K h^1_{\mu\nu}\big|$$

Now we have

$$\big|[(\widetilde{\Box}_g-\Box_0)\widehat{\mathcal{L}}_Z^I-\mathcal{L}_Z^I(\widetilde{\Box}_g-\Box_0)]h^1_{\mu\nu}\big|\lesssim\frac{\epsilon(1+q+)^{-\gamma}}{(1+t+r)^{1+\gamma-2\delta}(1+|q|)^{1-\gamma}}\sum_{|K|\le|I|}\big|\partial\widehat{\mathcal{L}}_Z^K h^1_{\mu\nu}\big|$$

Since

$$\mathcal{L}_Z^I(\widetilde{\Box}_g-\Box_0)]h^1_{\mu\nu}-[(\widetilde{\Box}_g-\Box_0)\widehat{\mathcal{L}}_Z^I-\mathcal{L}_Z^I(\widetilde{\Box}_g-\Box_0)]h^1_{\mu\nu}=(\widetilde{\Box}_g-\Box_0)\widehat{\mathcal{L}}_Z^I h^1_{\mu\nu}$$

we obtain in a similar way

$$\big|\mathcal{L}_Z^I(\widetilde{\Box}_g-\Box_0)]h^1_{\mu\nu}\big|\lesssim\frac{\epsilon(1+q+)^{-\gamma}}{(1+t+r)^{1+\gamma-2\delta}(1+|q|)^{1-\gamma}}\sum_{|K|\le|I|+1}\big|\partial\widehat{\mathcal{L}}_Z^K h^1_{\mu\nu}\big|\lesssim\frac{\epsilon^2(1+q+)^{-2\gamma}}{(1+t+r)^{2+\gamma-4\delta}(1+|q|)^{2-\gamma}}$$

Hence we finally get

$$\big|\Box_0\widehat{\mathcal{L}}_Z^I h^1_{\mu\nu}\big|\lesssim\frac{\epsilon(1+|q|)^{-1}}{1+t+r}\sum_{|J|\le|I|}\big|\partial\widehat{\mathcal{L}}_Z^J h\big|+\sum_{|K|\le|I|/2}\big|\partial\widehat{\mathcal{L}}_Z^K h\big|\sum_{|J|\le|I|-1}\big|\partial\widehat{\mathcal{L}}_Z^J h\big|$$
$$+\frac{\epsilon^2(1+q_+)^{-4\delta}}{(1+t+r)^{3-4\delta}(1+|q|)}+\frac{\epsilon^2(1+q+)^{-2\gamma}}{(1+t+r)^{2+\gamma-4\delta}(1+|q|)^{2-\gamma}}$$

Using Lemma 9.15 and Proposition 9.2 :

Lemma 9.18. *Let $D_t=\{(t,x),\ |t-|x||\le c_0t\}$, for some constant $0<c_0<1$ and let $\overline{w}(q)=(1+q_+)^{1+\gamma'}$, where $-1\le\gamma'<\gamma-2\delta$. Then*

$$(1+t+|x|)|\partial\widehat{\mathcal{L}}_Z^I h^1(t,x)\overline{w}(q)|\lesssim\epsilon+\int_0^t(1+\tau)\|(\Box_0\widehat{\mathcal{L}}_Z^I h^1)(\tau,.)\overline{w}\|_{L^\infty(D_\tau)}d\tau$$

Proposition 9.10. *If the weak energy bounds and initial bounds hold then we have for any $0\le\gamma'<\gamma-2\delta$, and $|I|=k\le N-5$, then there are constants c_k such that*

$$(9.47)\qquad (1+t+r)(1+q_+)^{1+\gamma'}|\partial\widehat{\mathcal{L}}_Z^I h^1|\lesssim c_k\epsilon(1+t)^{c_k\epsilon}.$$

The same estimates hold for h in place of h^1 if $\gamma'=0$

Proof. We shall prove this proposition by induction. Let $N_k(t)=(1+t)\sum_{|I|\le k}\|\Box_0\widehat{\mathcal{L}}_Z^I h^1(t,.)\overline{w}\|_{L^\infty(D_t)}$ and we show by induction that $N_k(t)\le c_k\epsilon(1+t)^{c_k\epsilon}$ The case $k=0$ has already been proven in Proposition

9.6. Suppose that we have the result until $k-1$ and we shall show it for $k \geq 1$. Recall that we have shown above that

$$|\Box_0 \widehat{\mathcal{L}}_Z^I h^1_{\mu\nu}| \lesssim \frac{\epsilon(1+|q|)^{-1}}{1+t+r} \sum_{|J|\leq |I|} |\partial \widehat{\mathcal{L}}_Z^J h| + \sum_{|K|\leq |I|/2} |\partial \widehat{\mathcal{L}}_Z^K h| \sum_{|J|\leq |I|-1} |\partial \widehat{\mathcal{L}}_Z^J h| + \frac{\epsilon^2(1+q_+)^{-4\delta}}{(1+t+r)^{3-4\delta}(1+|q|)} + \frac{\epsilon^2(1+q+)^{-2\gamma}}{(1+t+r)^{2+\gamma-4\delta}(1+|q|)^{2-\gamma}}$$

Moreover from previous Lemma we get the bound

$$(1+t+|x|)|\partial \widehat{\mathcal{L}}_Z^I h^1(t,x)\overline{w}(q)| \lesssim \epsilon + \int_0^t (1+\tau)\|(\Box_0 \widehat{\mathcal{L}}_Z^I h^1)(\tau,.)\overline{w}\|_{L^\infty(D_\tau)} d\tau$$

It follows that

$$N_k(t) \lesssim (1+t)\frac{\epsilon(1+|q|)^{-1}}{1+t+r} \times \frac{1}{1+t+r}\Big(\epsilon + \int_0^t (1+\tau)\|(\Box_0 \widehat{\mathcal{L}}_Z^I h^1)(\tau,.)\overline{w}\|_{L^\infty(D_\tau)} d\tau\Big) + \Big(\frac{1}{1+t+r}\Big(\epsilon + \int_0^t (1+\tau)\|(\Box_0 \widehat{\mathcal{L}}_Z^I h^1)(\tau,.)\overline{w}\|_{L^\infty(D_\tau)} d\tau\Big)\Big)^2$$

Hence

$$N_k(t) \lesssim \frac{\epsilon}{(1+t)}\Big(\epsilon + \int_0^t N_k(\tau)d\tau\Big) + \frac{1}{(1+t)^2}\Big(\epsilon + \int_0^t (1+\tau)\|(\Box_0 \widehat{\mathcal{L}}_Z^G h^1)(\tau,.)\overline{w}\|_{L^\infty(D_\tau)} d\tau\Big)^2$$

where $|G| = |I|-1$. Using Cauchy-Schwartz inequality on the second integral we finally get

$$N_k(t) \lesssim \epsilon + \int_0^t \frac{\epsilon}{1+\tau} N_k(\tau)d\tau + \int_0^t \frac{1}{1+\tau} N_{k-1}(\tau)^2 d\tau$$

Now with the induction assumption we have for some $c_k \geq 4c_{k-1}$

$$N_k(t) \leq c_k \int_0^t \frac{\epsilon}{1+\tau} N_k(\tau)d\tau + c_k\epsilon(1+t)^{2c_{k-1}\epsilon}$$

Applying Grönwall inequality gives

$$N_k(t) \leq c_k^2\epsilon(1+t)^{c_k\epsilon} + c_k\epsilon(1+t)^{2c_{k-1}\epsilon}$$

And hence we now write c_k such that

$$N_k(t) \leq c_k\epsilon(1+t)^{c_k\epsilon}$$

Inserting this into the integral of the Proposition gives the result.

$\square$

9.8 The energy estimate

It has been shown in [3] (p. 34/78) the following Lemma

Lemma 9.19. *Let ϕ be a solution of the wave equation $\widetilde{\Box}_g\phi = F$, with the metric g such that for $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$;*

$$(1+|q|)^{-1}|H|_{LL} + |\partial H|_{LL} + |\overline{\partial} H| \leq C\epsilon'(1+t)^{-1}$$
$$(1+|q|)^{-1}|H| + |\partial H| \leq C\epsilon'(1+t)^{-1/2} + (1+|q|)^{-1/2}(1+q_-)^{-\mu}$$

for some $\mu > 0$. Set

$$w = \begin{cases} (1+|r-t|)^{1+2\gamma} & r>t \\ 1+(1+|r-t|)^{-2\mu} & r\leq t \end{cases} \quad \text{and} \quad w' = \begin{cases} (1+2\gamma)(1+|r-t|)^{2\gamma} & r>t \\ 2\mu(1+|r-t|)^{-2\mu-1} & r\leq t \end{cases}$$

Then for any $0<\gamma\leq 1$, and $0<\epsilon'\leq \gamma/C_1$, we have

$$\int_{\Sigma_t} |\partial\phi|^2 w dx + \int_0^t \int_{\Sigma_\tau} |\overline{\partial}\phi|^2 w' dx d\tau \leq 8\int_{\Sigma_0} |\partial\phi|^2 w dx + \int_0^t \frac{C\epsilon}{1+\tau}\int_{\Sigma_t} |\partial\phi|^2 w dx d\tau + 16\int_0^t \Big(\int_{\Sigma_\tau} |F|^2 w dx\Big)^{1/2}\Big(\int_{\Sigma_\tau} |\partial\phi|^2 w dx\Big)^{1/2} d\tau$$

We can see that here the weight function gives a control of the tangential derivatives of ϕ which takes the form $\overline{\partial}\phi$.

Now let's remind our Theorem 3.1. We thus define

$$E_k(t) = \sum_{|I|\leq k}\int_{\Sigma_t} |\partial Z^I h^1|^2 w dx \quad \text{and} \quad S_k(t) = \sum_{|I|\leq k}\int_0^t\int_{\Sigma_t} |\partial Z^I h^1|^2 w' dx d\tau$$

In fact, the energy appearing here and in Theorem 3.1 really comes as a motivation from the energy inequality with weights. From the previous Lemma we get

$$E_0(t) + S_0(t) \leq 8E_0(0) + \int_0^t \frac{\epsilon}{1+\tau}E_0(\tau) + \|F(\tau,.)w^{1/2}\|_{L^2}E_0(\tau)^{1/2}d\tau,$$

where $|F| = |F_{\mu\nu}(h)(\partial h, \partial h) + \widehat{T}_{\mu\nu} - \widetilde{\Box}_g h^0_{\mu\nu}|$. Now, from Lemma 9.11 we get

$$|F_{\mu\nu}(h)(\partial h,\partial h) - \not{P}_{\mu\nu}(\partial h,\partial h)| \lesssim |\overline{\partial}h||\partial h| + |h||\partial h|^2$$

From the definition of $\not{P}_{\mu\nu}$, Proposition 9.6 and Proposition 9.2 we have

$$|F_{\mu\nu}(h)(\partial h,\partial h)| \lesssim \frac{\epsilon|\partial h|}{(1+t+r)(1+q_+)}$$

Writing $h = h^0 + h^1$ and for $j = 0,1$:

$$F^j = \frac{\epsilon|\partial h^j|}{(1+t+r)(1+q_+)}$$

Hence we have

$$\|F^1(t,.)w^{1/2}\|_{L^2} \lesssim \epsilon(1+t)^{-1}\|\partial h^1(t,.)w^{1/2}\| = \epsilon(1+t)^{-1}E_0(t)^{1/2}$$

and from

$$F^0 \lesssim \frac{M\epsilon}{(1+q_+)(1+t+r)^3}$$

we get

$$\|F^0(t,.)w^{1/2}\|_{L^2} \lesssim \frac{M\epsilon}{(1+t)^{2-\gamma}}$$

We now need to estimate the L^2 norm of $\widetilde{\Box}_g h^0_{\mu\nu}$. In order to do that recall the Hardy-type inequality : For any $-1 \leq a \leq 1$ and any $\phi \in C_0^\infty(\mathbb{R}^3)$:

$$\int \frac{|\phi|^2}{(1+|t-r|)^2}\frac{w}{(1+t+r)^{1-a}}dx \lesssim \int |\partial\phi|^2 \frac{w}{(1+t+r)^{1-a}}dx$$

Moreover we have

$$|\widetilde{\Box}_g h^0_{\mu\nu}| \leq |[\widetilde{\Box}_g - \Box_0]h^0_{\mu\nu}| + |[\Box_0 - \Box]h^0_{\mu\nu}| + |\Box h^0_{\mu\nu}|$$

From the definition of the operator and using

$$\left|\mathcal{L}^I_Z(\widetilde{\Box}_g - \Box_0)h^0_{\mu\nu}\right| \lesssim \frac{M}{(1+t+r)^3}\sum_{|J|\leq|I|}\left|\widehat{\mathcal{L}}^J_Z H_1\right|$$

we get

$$|\widetilde{\Box}_g h^0_{\mu\nu}| \leq \frac{C_0|H_1|M}{(1+t+r)^3} + \frac{M^2\chi'\left(\frac{r}{t+1}\right)}{(1+t+r)^4} + \frac{M\chi'\left(\frac{r}{t+1}\right)}{(1+t+r)^3}$$

Using the previous bound in the Hardy's inequality gives

$$\|\widetilde{\Box}_g h^0_{\mu\nu}w^{1/2}\|_{L^2} \leq CM(1+t)^{-2}\|\partial H_1(t,.)w^{1/2}\|_{L^2} + M_0M(1+t)^{-3/2}$$

where M_0 is a universal constant. Hence

$$E_0(t) \leq 8E_0(0) + C'\epsilon\int_0^t \frac{E_0(\tau)}{1+\tau}d\tau + C'\epsilon\int_0^t \frac{ME_0(\tau)^{1/2}}{(1+\tau)^{2-\gamma}}d\tau + 16M_0\int_0^t \frac{ME_0(\tau)^{1/2}}{(1+\tau)^{3/2}}d\tau + 16\int_0^t \|\widehat{T}(\tau,.)\|_{L^2}E_0(\tau)^{1/2}d\tau$$

9.9 Higher order L^2 Energy estimates

In this subsection we make the assumption on ϵ : $c_{k'}\epsilon \leq \delta$ where the constants $c_{k'}$ have been defined in Proposition 9.10. We write :

$$F_1^{kj} = \frac{\epsilon}{(1+t+r)(1+q_+)} \sum_{|J|\leq k} \left|\partial \widehat{\mathcal{L}}_Z^J h^j\right|, \qquad F_2^{kj} = \frac{c_{k'}\epsilon(1+t)^{c_{k'}\epsilon}}{(1+t+r)(1+q_+)} \sum_{|J|\leq k-1} \left|\partial \widehat{\mathcal{L}}_Z^J h^j\right|$$

$$F_3^{kj} = \frac{\epsilon^2(1+t)^{4\delta}(1+q_+)^{-4\delta}}{(1+t+r)^2(1+|q|)^2} \sum_{|J|\leq |I|} \left|\widehat{\mathcal{L}}_Z^J h^j\right|, \qquad F_4^{kj} = \frac{\epsilon(1+t)^{2\delta}(1+q_+)^{-2\delta}}{(1+t+r)(1+|q|)} \sum_{|J|\leq |I|} \left|\overline{\partial} \widehat{\mathcal{L}}_Z^J h^j\right|$$

From the inequality (9.40) and with $k = |I|$, $k' = [k/2]+1$ we have :

$$\left|\mathcal{L}_Z^I F_{\mu\nu}(h)(\partial h, \partial h)\right| \lesssim F_1^{k0} + F_1^{k1} + F_2^{k0} + F_2^{k1} + F_3^{k0} + F_3^{k1} + F_4^{k0} + F_4^{k1}$$

Since

$$\sum_{|I|\leq k} |Z^I K| \lesssim \sum_{|I|\leq k} |\widehat{\mathcal{L}}_Z^I K| \lesssim \sum_{|I|\leq k} |Z^I K|$$

we have the following L^2 estimates :

$$\Big(\int |F_1^{k1}|^2 w dx\Big)^{1/2} \lesssim \frac{\epsilon}{1+t} E_k(t)^{1/2} \qquad \Big(\int |F_2^{k1}|^2 w dx\Big)^{1/2} \lesssim \frac{c_{k'}\epsilon(1+t)^{c_{k'}\epsilon}}{1+t} E_{k-1}(t)^{1/2}$$

Now from Proposition 9.2

$$|F_1^{k0}| \lesssim \frac{\epsilon M}{(1+t+r)^3(1+q_+)} \qquad F_2^{k0} \lesssim \frac{\epsilon M(1+t)^{c_{k'}\epsilon}}{(1+t+r)^3(1+q_+)}$$

we get the L^2 estimates :

$$\Big(\int |F_1^{k0}|^2 w dx\Big)^{1/2} \lesssim \frac{\epsilon M}{(1+t)^{2-\gamma}} \qquad \Big(\int |F_2^{k0}|^2 w dx\Big)^{1/2} \lesssim \frac{c_{k'}\epsilon M}{(1+t)^{2-\gamma-c_{k'}\epsilon}}$$

The inequality for F_3^{k0} and F_4^{k0} can be estimated in a similar way and we get

$$\Big(\int |F_3^{k0}|^2 w dx\Big)^{1/2} \lesssim \frac{\epsilon^2 M}{(1+t)^{2-4\delta}}$$

$$\Big(\int |F_4^{k0}|^2 w dx\Big)^{1/2} \lesssim \frac{\epsilon M}{(1+t)^{3-2\gamma}}$$

Using Hardy's inequality for F_3^{k1} we get

$$\Big(\int |F_3^{k1}|^2 w dx\Big)^{1/2} \lesssim \frac{\epsilon}{(1+t)^{2-4\delta}} E_k(t)^{1/2}$$

For the term F_4^{k1} we first use Cauchy-Schwarz inequality and we get

$$\int |F_4^{k1}|^2 w dx \lesssim \int \frac{\epsilon^2(1+t)^{4\delta}(1+q_+)^{-4\delta}}{(1+t+r)^2(1+|q|)^2} \sum_{|J|\leq |I|} \left|\overline{\partial} \widehat{\mathcal{L}}_Z^J h^1\right|^2 w dx$$

We then first bound the fraction that appears in the integral by taking $r = 0$ in the denominator and since $w(1+q_+)^{-4\delta}(1+|q|)^2 \lesssim w'$ we get

$$\int |F_4^{k1}|^2 w dx \lesssim \frac{\epsilon^2}{(1+t)^{2-4\delta}} \int \sum_{|J|\leq |I|} \left|\overline{\partial} \widehat{\mathcal{L}}_Z^J h^1\right|^2 w' dx$$

Now with the inequality of arithmetic and geometric means we get

$$\int \Big(\int |F_4^{k1}|^2 w dx\Big)^{1/2} E_k(\tau)^{1/2} \leq \epsilon S_k(t) + C^2 \int_0^t \frac{\epsilon E_k(\tau)}{(1+\tau)^{2-4\delta}} d\tau$$

Since $\delta \leq 1/4$ we finally get

(9.48)
$$\int_0^t \int |\mathcal{L}_Z^I F_{\mu\nu}||\partial \mathcal{L}_Z^I h^1| w dx d\tau \leq C' \int_0^t \Big(\frac{\epsilon E_k(\tau)^{1/2}}{1+\tau} + \frac{c_{k'}\epsilon(1+\tau)^{c_{k'}\epsilon}}{1+\tau} E_{k-1}(\tau)^{1/2} + \frac{\epsilon M}{(1+t)^{2-\gamma-c_{k'}\epsilon}}\Big) E_k(\tau)^{1/2} d\tau + \epsilon S_k(t)$$

Now, we want an estimate of the wave operator applied to h^0. From

$$\left|\mathcal{L}_Z^I(\widetilde{\Box}_g - \Box_0)h^0_{\mu\nu}\right| \lesssim \frac{M}{(1+t+r)^3}\sum_{|J|\leq|I|}\left|\widehat{\mathcal{L}}_Z^J H_1\right|$$

and using Hardy's inequality we get

$$\|(\mathcal{L}_Z^I(\widetilde{\Box}_g - \Box_0)h^0(t,.))w^{1/2}\|_{L^2} \lesssim \frac{M}{(1+t)^2}\sum_{|I|\leq k}\|\partial\widehat{\mathcal{L}}_Z^I H_1(t,.)w^{1/2}\|_{L^2}$$

Hence, it remains to estimate $\|\partial\widehat{\mathcal{L}}_Z^I H_1(t,.)w^{1/2}\|_{L^2}$. In order to do so we recall that we have $H_1 = -h^1 + K(h) - h^0 - H_0$ where $K(h) = O(h^2)$. Moreover we can estimate the factors with h^1 after using Hardy's inequality and express term in term of the Energy $E_k(t)$. Since h^0 is explicit we have

$$\sum_{|I|\leq k}\|\partial\widehat{\mathcal{L}}_Z^I H_1(t,.)w^{1/2}\|_{L^2} \lesssim \epsilon M + E_k(t)^{1/2}$$

It follows

$$\|(\mathcal{L}_Z^I(\widetilde{\Box}_g - \Box_0)h^0(t,.))w^{1/2}\|_{L^2} \lesssim \frac{M}{(1+t)^2}E_k(t)^{1/2}$$

Moreover, since

$$\left|\mathcal{L}_Z^I\Box_0 h^0_{\mu\nu}\right| \lesssim \frac{M}{(1+t+r)^3}\chi'\left(\frac{r}{1+t}\right)$$

we get

$$\sum_{|I|\leq k}\|(\mathcal{L}_Z^I\Box_0 h^0)(t,.)w^{1/2}\|_{L^2} \lesssim \epsilon M_k^2\frac{M^2}{(1+t)^3}$$

for some universal constant M_k.

It remains to estimate the commutator. With (9.44) and (9.45) we get

$$\left|[\widetilde{\Box}_g\widehat{\mathcal{L}}_Z^I - \mathcal{L}_Z^I\widetilde{\Box}_g]h^1_{\mu\nu}\right| \lesssim F_5^k + F_6^k + F_6^k$$

where

$$F_5^k = \frac{\epsilon}{(1+t)}\sum_{|K|\leq|I|}\left|\partial\widehat{\mathcal{L}}_Z^K h^1_{\mu\nu}\right|$$

and

$$F_6^k = \frac{\epsilon(1+q_+)^{-\gamma}}{(1+t+r)^{2-2\gamma}}\sum_{|J|\leq|I|}\frac{\left|\widehat{\mathcal{L}}_Z^J H_1\right|}{1+|q|} \qquad F_7^k = \frac{\epsilon(1+t)^{2\delta}(1+q_+)^{-\gamma}}{(1+t+r)(1+|q|)}\sum_{|J|\leq|I|}\frac{\left|(\widehat{\mathcal{L}}_Z^J H_1)_{LL}\right|}{1+|q|}\chi\Big(\frac{r}{t+1}\Big)$$

For F_5^k and F_6^k we have in a similar way

$$\Big(\int|F_5^k|^2 w dx\Big)^{1/2} \lesssim \frac{\epsilon}{1+t}E_k(t)^{1/2}$$

$$\Big(\int|F_5^k|^2 w dx\Big)^{1/2} \lesssim \frac{\epsilon}{(1+t)^{2-2\delta}}E_k(t)^{1/2}$$

For F_7^k we shall use the following Corollary which has been shown in [3] (p. 68/80) :

Corollary 9.1. Let $\gamma > 0$ and $\mu > 0$. Then for any $-1 \leq a \leq 1$ and any $\phi \in C_0^\infty(\mathbb{R}^3)$, in in addition $a < 2\min(\gamma,\mu)$, we have :

$$\int\frac{|\phi|^2}{(1+|q|)^2}\frac{(1+|q|)^{-a}}{(1+t+|q|)^{1-a}}\frac{w}{(1+q_-)^{2\mu}}dx \lesssim \int|\partial\phi|^2\min(w', \frac{w}{(1+t+|q|)^{1-a}})dx$$

Now we estimate the last term F_7^k. From Lemma 9.10 we have

$$|\partial_q\widehat{\mathcal{L}}_Z^I H|_{LT} + |\partial_q \cancel{tr}\widehat{\mathcal{L}}_Z^I H| \lesssim |\overline{\partial}\widehat{\mathcal{L}}_Z^I H| + \sum_{|J|+|K|\leq|I|}|\widehat{\mathcal{L}}_Z^J h||\partial\widehat{\mathcal{L}}_Z^K h|,$$

$$|\partial_q\widehat{\mathcal{L}}_Z^I H_1|_{LT} + |\partial_q \cancel{tr}\widehat{\mathcal{L}}_Z^I H_1| \lesssim |\overline{\partial}\widehat{\mathcal{L}}_Z^I H_1| + \sum_{|J|+|K|\leq|I|}|\widehat{\mathcal{L}}_Z^J h||\partial\widehat{\mathcal{L}}_Z^K h| + M\left|\chi'\left(\frac{r}{t+1}\right)\right|(1+t+r)^{-2},$$

First, using the fact that $(1+t) \leq (1+t+r)$ and that χ is a smooth cut-off function we write

$$\int |F_7^k|^2 w dx \lesssim \frac{\epsilon^2}{1+\tau} \sum_{|J|\leq|I|} \int \frac{(1+q_+)^{-2\gamma}}{(1+\tau+r)^{1-4\delta}(1+|q|)^2} \frac{\left|(\widehat{\mathcal{L}}_Z^J H_1)_{LL}\right|^2}{(1+|q|)^2} w dx$$

Using the previous corollary we get

$$\int |F_7^k|^2 w dx \lesssim \frac{\epsilon^2}{1+\tau} \sum_{|J|\leq|I|} \int \left|\partial(\widehat{\mathcal{L}}_Z^J H_1)_{LL}\right|^2 \min\Big(w', \frac{w}{(1+\tau+|q|)^{1-4\delta}}\Big) dx$$

With the inequalities from Lemma 9.10 stated above and the inequality of arithmetic and geometric means we get

$$\begin{aligned}
&\int |F_7^k|^2 w dx \\
&\lesssim \frac{\epsilon^2}{1+\tau}\Big(\sum_{|J|\leq|I|} \int \left|\overline{\partial}(\widehat{\mathcal{L}}_Z^J H_1)_{LL}\right|^2 w' dx + \sum_{|J|+|K|\leq|I|} \int \frac{w}{(1+t+r)^{1-4\delta}} \left|\widehat{\mathcal{L}}_Z^J h\right|^2 \left|\partial \widehat{\mathcal{L}}_Z^K h\right|^2 dx + \int_{|x|\leq 3\tau/4} \frac{M^2 w' dx}{(1+\tau)^4}\Big) \\
&\lesssim \frac{\epsilon^2}{1+\tau} \sum_{|J|\leq|I|} \int \left|\overline{\partial}(\widehat{\mathcal{L}}_Z^J H_1)_{LL}\right|^2 w' dx + \frac{\epsilon^4}{1+\tau} \sum_{|J|\leq|I|} \int \frac{(1+q_+)^{-4\delta}}{(1+t+r)^{3-8\delta}} \Big(\left|\partial \widehat{\mathcal{L}}_Z^J h\right|^2 + \frac{\left|\widehat{\mathcal{L}}_Z^J h\right|^2}{(1+|q|)^2}\Big) w dx + \frac{\epsilon^2 M^2}{(1+\tau)^{3+2\mu}}
\end{aligned}$$

Now, we write $h = h^0 + h^1$. Using the fact that

$$w(t,x) = \begin{cases} (1+|r-t|)^{1+2\gamma} & r > t \\ 1 + (1+|r-t|)^{-2\mu} & r \leq t \end{cases}$$

,splitting the integral in two parts, using the definition of h^0 and that the Lie derivatives and the vector fields are comparable we get :

$$\sum_{|J|\leq|I|} \int \frac{(1+q_+)^{-4\delta}}{(1+t+r)^{3-8\delta}} \Big(\left|\partial \widehat{\mathcal{L}}_Z^J h^0\right|^2 + \frac{\left|\widehat{\mathcal{L}}_Z^J h^0\right|^2}{(1+|q|)^2}\Big) w dx \lesssim \frac{M^2}{(1+\tau)^{3-2\gamma-4\delta}}$$

From Hardh's inequality we have

$$\sum_{|J|\leq|I|} \int \frac{(1+q_+)^{-4\delta}}{(1+t+r)^{3-8\delta}} \Big(\left|\partial \widehat{\mathcal{L}}_Z^J h^1\right|^2 + \frac{\left|\widehat{\mathcal{L}}_Z^J h^1\right|^2}{(1+|q|)^2}\Big) w dx \lesssim \frac{1}{(1+\tau)^{3-8\delta}} \sum_{|J|\leq|I|} \int \left|\partial \widehat{\mathcal{L}}_Z^J h^1\right|^2 w dx$$

Hence

$$\int |F_7^k|^2 w dx \lesssim \frac{\epsilon^2}{1+\tau} \sum_{|J|\leq|I|} \int \left|\overline{\partial}(\widehat{\mathcal{L}}_Z^J H_1)_{LL}\right|^2 w' dx + \frac{\epsilon^2}{(1+\tau)^3}\Big(\sum_{|J|\leq|I|} \epsilon^2 \int |\partial \widehat{\mathcal{L}}_Z^J h^1|^2 w dx + M^2\Big) + \frac{\epsilon^4 M^2}{(1+\tau)^{4-2\gamma-4\delta}}$$

It follows that

$$\begin{aligned}
&\int \Big(\int |F_7^k|^2 w dx\Big)^{1/2} E_k(\tau)^{1/2} d\tau \\
&\qquad\leq \epsilon S_k(t) + \int_0^t \frac{C'\epsilon}{1+\tau} E_k(\tau) d\tau + \int_0^t \Big(\frac{C'\epsilon}{(1+\tau)^{3/2}} + \frac{C'\epsilon^2}{(1+\tau)^{2-\gamma-2\delta}}\Big) M E_k(\tau)^{1/2} d\tau
\end{aligned}$$

Hence from

$$\left|[\widetilde{\Box}_g \widehat{\mathcal{L}}_Z^I - \mathcal{L}_Z^I \widetilde{\Box}_g] h^1_{\mu\nu}\right| \lesssim F_5^k + F_6^k + F_6^k$$

we get

$$\begin{aligned}
(9.49)\quad &\int_0^t \int \left|[\widetilde{\Box}_g \widehat{\mathcal{L}}_Z^I - \mathcal{L}_Z^I \widetilde{\Box}_g] h^1_{\mu\nu}\right| |\partial \mathcal{L}_Z^I h^1| w dx d\tau \\
&\qquad\leq \epsilon S_k(t) + \int_0^t \frac{C'\epsilon}{1+\tau} E_k(\tau) d\tau + \int_0^t \Big(\frac{C'\epsilon}{(1+\tau)^{3/2}} + \frac{C'\epsilon^2}{(1+\tau)^{2-\gamma-2\delta}}\Big) M E_k(\tau)^{1/2} d\tau
\end{aligned}$$

Theorem 9.1. *Suppose that $N \geq 9$. Then there is a $\epsilon_N > 0$ and constants $C'''_N, d_1, ..., d_N$ depending only on N, C_N from (9.1),c,K and a lower positive bound for $\min(\gamma, 1-\gamma)$, such that for all $t \in [0, T_*]$ and $\epsilon < \epsilon_N$,*

$$Q_k(t) \leq 8Q_k(0) + M_k M + C'''_N \epsilon \int_0^t \frac{Q_k(\tau)}{1+\tau} + \frac{Q_{k-1}(\tau)}{(1+\tau)^{1-d_k\epsilon}} d\tau + M_k \sum_{|I|\leq k} \int_0^t \|Z^I \widehat{T}(\tau,.)\|_{L^2} d\tau,$$

where $Q_k(t) := \sup_{0\leq\tau\leq t} E_k(\tau)^{1/2}$ and $Q_{-1}(0) = 0$, and $M_1, ..., M_k$ are universal constants.

Proof. We shall prove this theorem by induction. The case $k = 0$ has been shown. Suppose that the results holds for some $k-1$ and we shall prove it for k. To understand the proof I will write the ingredients that we will need here. First the definitions

$$E_k(t) = \sum_{|I|\leq k} \int_{\Sigma_t} |\partial Z^I h^1|^2 w dx \quad \text{and} \quad S_k(t) = \sum_{|I|\leq k} \int_0^t \int_{\Sigma_t} |\partial Z^I h^1|^2 w' dx d\tau$$

Then the energy inequality (Lemma 9.19)

$$\int_{\Sigma_t} |\partial\phi|^2 w dx + \int_0^t \int_{\Sigma_\tau} |\overline{\partial}\phi|^2 w' dx d\tau \leq 8 \int_{\Sigma_0} |\partial\phi|^2 w dx + \int_0^t \frac{C\epsilon}{1+\tau} \int_{\Sigma_t} |\partial\phi|^2 w dx d\tau$$
$$+ 16 \int_0^t \Big(\int_{\Sigma_\tau} |F|^2 w dx\Big)^{1/2} \Big(\int_{\Sigma_\tau} |\partial\phi|^2 w dx\Big)^{1/2} d\tau$$

The inequalities (9.48) and (9.49)

$$\int_0^t \int |\mathcal{L}_Z^I F_{\mu\nu}||\partial \mathcal{L}_Z^I h^1| w dx d\tau \leq C' \int_0^t \Big(\frac{\epsilon E_k(\tau)^{1/2}}{1+\tau} + \frac{c_{k'}\epsilon(1+\tau)^{c_{k'}\epsilon}}{1+\tau} E_{k-1}(\tau)^{1/2} + \frac{\epsilon M}{(1+t)^{2-\gamma-c_{k'}\epsilon}}\Big) E_k(\tau)^{1/2} d\tau + \epsilon S_k(t)$$

$$\int_0^t \int |[\widetilde{\Box}_g \widehat{\mathcal{L}}_Z^I - \mathcal{L}_Z^I \widetilde{\Box}_g] h^1_{\mu\nu}||\partial \mathcal{L}_Z^I h^1| w dx d\tau$$
$$\leq \epsilon S_k(t) + \int_0^t \frac{C'\epsilon}{1+\tau} E_k(\tau) d\tau + \int_0^t \Big(\frac{C'\epsilon}{(1+\tau)^{3/2}} + \frac{C'\epsilon^2}{(1+\tau)^{2-\gamma-2\delta}}\Big) M E_k(\tau)^{1/2} d\tau$$

The equation

$$\widetilde{\Box}_g \widehat{\mathcal{L}}_Z^I h_{\mu\nu} = [\widetilde{\Box}_g \widehat{\mathcal{L}}_Z^I - \mathcal{L}_Z^I \widetilde{\Box}_g] h_{\mu\nu} + \mathcal{L}_Z^I F_{\mu\nu}(H)(\partial h, \partial h) + \mathcal{L}_Z^I T_{\mu\nu}$$

Finally the fact the vector fields and Lie derivatives are comparable

$$\sum_{|I|\leq k} |Z^I K| \lesssim \sum_{|I|\leq k} |\widehat{\mathcal{L}}_Z^I K| \lesssim \sum_{|I|\leq k} |Z^I K|$$

From all these ingredients, applying the energy inequality to $Z^I h^1$ we get

$$E_k(t) + S_k(t) \leq 8E_k(0) + 32\epsilon S_k(t) + 16M_k \int_0^t \frac{M E_k(\tau)^{1/2}}{(1+\tau)^{3/2}} d\tau + C''\epsilon \int_0^t \Big(\frac{\epsilon E_k(\tau)}{1+\tau} + \frac{c_{k'}\epsilon(1+\tau)^{c_{k'}\epsilon}}{1+\tau} E_{k-1}(\tau)\Big) d\tau$$
$$+ C'' \int_0^t \Big(\frac{\epsilon^2}{(1+\tau)^{2-\gamma-\max(2\delta, c_{k'}\epsilon)}} + \frac{\epsilon}{(1+\tau)^{3/2}}\Big) M E_k(\tau)^{1/2} d\tau + 16 \int_0^t \Big(\sum_{|I|\leq k} \|Z^I \widehat{T}(\tau,.)\|^2_{L^2}\Big)^{1/2} E_k(\tau)^{1/2} d\tau$$

We choose ϵ small enough os that $32\epsilon \leq 1$ to take off $S_k(t)$ from the inequality and so that $c_{k'}\epsilon \leq 2\delta$ and so that $C''\epsilon \leq M_k$. Theorem 9.1 follows.

□

10 Proof of Theorem 3.1

This section is dedicated to the proof of our main theorem. We follow exactly the path of the paper from Hans Lindblad and Martin Taylor [1]. One gives T_* the supremum of all times T_1 such that a solution of the reduced Einstein-Vlasov system attaining the given data exists for all $t \in [0, T_1]$ and satisfies the bounds :

$$E_N(t)^{1/2} \leq C_N \epsilon (1+t)^\delta, \qquad \sum_{|I|\leq N-1} \|Z^I \hat{T}(t,.)\|_{L^1} \leq C_N \epsilon \tag{10.1}$$

for all $t \in [0, T_1]$, where $\delta > 0$ is such that $\delta < \gamma < 1 - 8\delta$ and C_N is a fixed large constant to be detemined. The set of such T_1 is non empty by the local existence theorem of Choquet-Bruhat and hence $T_* > 0$ The main objective is thus to use the Einstein equations to prove that the bounds in fact hold with better constants, provided the initial data are sufficiently small. We suppose $T_* < \infty$.

We have the following bounds on h :

$$|h(t,x)| \leq \frac{C'_N \epsilon (1+t)^{2\delta}}{(1+t+r)(1+q_+)^{2\delta}}, \qquad \sum_{|I|+|J| \leq N-4} |\partial^I Z^J \partial h(t,x)| \leq \frac{C'_N \epsilon (1+t)^{2\delta}}{(1+t+r)(1+|q|)(1+q_+)^{2\delta}} \tag{10.2}$$

Hence the assumptions for proposition 4.1 are verified and provided ϵ is small enough we get

$$\mathrm{supp}(\widehat{T}^{\mu\nu}) \subset \{(t,x) \mid |x| \leq ct + K\}$$

for some $0 < c < 1$, $K \geq 0$.

Hence we have from Proposition 9.6 and Proposition 9.10, and Theorem 9.1 :

$$Q_k(t) \leq 8Q_k(0) + M_k M + C'''_N \epsilon \int_0^t \frac{Q_k(\tau)}{1+\tau} + \frac{Q_{k-1}(\tau)}{(1+\tau)^{1-d_k\epsilon}} d\tau + M_k \sum_{|I| \leq k} \int_0^t \|Z^I \widehat{T}(\tau, .)\|_{L^2} d\tau, \tag{10.3}$$

where $Q_k(t) := \sup_{0 \leq \tau \leq t} E_k(\tau)^{1/2}$ and $Q_{-1}(0) = 0$, and $M_0, ..., M_k$ are universal constants for each $k = 0, 1, ..., N$ and for all $t \in [0, T_*]$ where $Q_{-1}(t) = 0$

Moreover, (10.2) imply that

$$\sum_{|I| \leq N-4} |Z^I \Gamma(t,x)| \leq \frac{C'_N \epsilon}{(1+t)^{1+a}}$$

for $t \in [0, T_*]$ and $|x| \leq ct + K$ with $a = 2 - 2\delta$. The assumptions on theorem 3.2 are therefore satistifed so

$$\begin{aligned}
\sum_{|I| \leq k} \|Z^I T(t,.)\|_{L^2} &\lesssim \frac{\mathcal{V}_k(1+D_N)\epsilon}{(1+t)^{3/2}} + D_N \mathbb{D}_{\lfloor \frac{k}{2} \rfloor + 1} \Big(\frac{E_{k-1}(t)^{1/2}}{(1+t)^{1+a}} + \frac{1}{(1+t)^{3/2}} \int_0^t \frac{E_k(s)^{1/2}}{(1+s)^{1/2}} ds \Big) \\
&\lesssim \frac{\mathcal{V}_k(1+D_N)\epsilon}{(1+t)^{3/2}} + D_N \mathbb{D}_{\lfloor \frac{k}{2} \rfloor + 1} \frac{Q_k(t)}{(1+t)}
\end{aligned}$$

for all $k = 0, 1, ..., N$ and for all $t \in [0, T_*]$, where the constant D_N depends on C_N. From Lemma 9.8 we have

$$\sum_{|I| \leq N-4} \|Z^I \widehat{T}(t,.)\|_{L^\infty} \leq \frac{C_N \epsilon}{(1+t)^3}$$

From the support condition on $\widehat{T}$ we define the function ψ such that $\psi(t,x)$ is the indicator function of the set $\{(t,x) \mid |x| \leq ct + K\}$. Recall that $\widehat{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} Tr_g T$. Writing $g = m + h^0 + h^1$ we have $\widehat{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{2} m^{\mu\nu} Tr_g T - \frac{1}{2} h^{0\mu\nu} Tr_g T - \frac{1}{2} h^{1\mu\nu} Tr_g T$ Hence we have

$$\begin{aligned}
\sum_{|I| \leq k} \|Z^I \widehat{T}(t,.)\|_{L^2} \lesssim \sum_{|I| \leq k} \|Z^I T(t,.)\|_{L^2} &+ \sum_{|J| \leq \lfloor k/2 \rfloor} \|Z^I T(t,.)\|_{L^\infty} \sum_{1 \leq |I| \leq k} \|\psi Z^I h^1(t,.)\|_{L^2} \\
&+ \sum_{|J| \leq \lfloor k/2 \rfloor + 1} \|\psi Z^I h^1(t,.)\|_{L^\infty} \sum_{|I| \leq k} \|Z^I T(t,.)\|_{L^2}
\end{aligned}$$

Hence

$$\sum_{|I| \leq k} \|Z^I T(t,.)\|_{L^2} \lesssim \frac{\mathcal{V}_k(C+D_N)\epsilon}{(1+t)^{3/2}} + D_N \mathbb{D}_{\lfloor \frac{k}{2} \rfloor + 1} \frac{Q_k(t)}{(1+t)} + D_N \epsilon \frac{Q_k(t)}{(1+t)}$$

for $k = 0, 1, ..., N$ where the constant C is independent of C_N and the constant D_n depends on C_N. Inserting into (10.3) and using the fact that,

$$Q_N(0) + \mathbb{D}_{\lfloor N/2 \rfloor + 1} + \mathcal{V}_N + M < \epsilon$$

and changing M_k and C'''_N if necessary gives,

$$Q_k(t) \leq M_k \epsilon + C'''_N \epsilon \int_0^t \frac{Q_k(\tau)}{1+\tau} + \frac{Q_{k-1}(\tau)}{(1+\tau)^{1-d_k\epsilon}} d\tau, \tag{10.4}$$

for $k = 0, 1, ..., N$ From an induction and using the previous bound along with the Grönwall inequality it follows that

$$Q_k(t) \leq (M_0 + ... + M_k)\epsilon(1+t)^{(d_1+...+d_k+(k+1)C'''_N)\epsilon} \tag{10.5}$$

Moreover, theorem 3.2 implies that,

$$\sum_{|I|\leq N-1} \|Z^I\widehat{T}(t,.)\|_{L^1} \lesssim \epsilon(C+\epsilon D_N) + \epsilon D_N \frac{Q_{N-2}(t)}{(1+t)^{1/2-2\delta}} + \epsilon D_N \int_0^t \frac{Q_N(s)}{(1+s)^{3/2-2\delta}} ds,$$

where the constant C is independent of C_N and the constant D_N depends on C_N. Inserting the above bound for Q_N gives

$$\sum_{|I|\leq N-1} \|Z^I\widehat{T}(t,.)\|_{L^1} \lesssim C\epsilon + D_N\epsilon^2$$

provided ϵ is small enough for some new C, D_N as above. As above,

$$\begin{aligned}\sum_{|I|\leq N-1} \|Z^I\widehat{T}(t,.)\|_{L^1} \lesssim \sum_{|I|\leq N-1} \|Z^I T(t,.)\|_{L^1} &+ \sum_{|J|\leq \lfloor N/2\rfloor} \|Z^I T(t,.)\|_{L^\infty} \sum_{1\leq|I|\leq N-1} \|\psi Z^I h^1(t,.)\|_{L^1} \\ &+ \sum_{|J|\leq \lfloor N/2\rfloor+1} \|\psi Z^I h^1(t,.)\|_{L^\infty} \sum_{|I|\leq N-1} \|Z^I T(t,.)\|_{L^1}\end{aligned}$$

And with Cauchy-Schwarz inequality we get

$$\sum_{1\leq|I|\leq N-1} \|\psi Z^I h^1(t,.)\|_{L^1} \leq (1+t)^{3/2}\frac{1}{1+t} E_{|I|}(t)^{1/2}$$

and it follows that

$$\sum_{|I|\leq N-1} \|Z^I\widehat{T}(t,.)\|_{L^1} \leq (C+\epsilon C'''_N)(C\epsilon + D_N\epsilon^2) + \frac{C+\epsilon C_N}{(1+t)^3} Q_N(t)$$

and so,

$$\sum_{|I|\leq N-1} \|Z^I\widehat{T}(t,.)\|_{L^1} \leq C'\epsilon + D'_N\epsilon^2 \tag{10.6}$$

where D'_N depends on C_N and C' does not. It follows from the bound (10.5) with $k = N$ and the bound (10.6), provided the constant C_N is chosen so that $C_N \geq \max(2(M_0 + ... + M_N), 4C)$ and ϵ is chosen so that $\epsilon < \min(\frac{\delta}{2}(d_1 + ... + d_N + (N+1)C'''_N)^{-1}, \frac{C_N}{4D'_N})$, that the bounds

$$E_N(t)^{1/2} \leq \frac{C_N}{2}\epsilon(1+t)^{\delta/2}, \qquad \sum_{|I|\leq N-1} \|Z^I\widehat{T}(t,.)\|_{L^1} \leq \frac{C_N}{2}\epsilon$$

hold for all $t \in [0, T_*]$. Appealing once again to the local existence theorem, this contradicts the maximality of T_* and hence the solution exists and the estimates hold for all $t \in [0, \infty)$.

11 Conclusion

More than a hundred years have passed since 1908 when Hermann Minkowski gave a four-dimensional formulation of special relativity according to which space and time are united into an inseparable four-dimensional entity. Minkowski metric is on the basis of empty space solution of Einstein solution. It is the flat spacetime manifold $\mathbb{R}^4$. This report was dedicated to the stability of Minkowski space but one could ask if this spacetime is an accurate representation of the reality. Indeed, as we've seen, Einstein's theory of general relativity details how massive objects warp the spacetime around them and thus, locally, spacetime is curved around every object with mass. A measure of the curvature of the spacetime is given by the density parameter Ω which is defined as the ratio of the actual density to the critical density of the Friedmann universe (which is a model universe in motion). If $\Omega = 1$ the universe is considered flat, $\Omega > 1$ would indicate that the universe has a positive curvature whereas $\Omega < 1$ leads to a negative curvature. Measurements from the Wilkinson Microwave Anisotropy Probe (WMAP) have shown the observable universe to have a density very close to the critical density within a 0.4% margin error. Hence, if the assumption on the work previously done are true, we should stay close to this space for a long time.

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